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A Note on the Order Bidual of f-Algebras

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Abstract

The paper deals with the Arens Multiplication which we accomplished in four steps in the order bidual $X^{\sim\sim}$. It is shown that if f is an element of order dual $X^{\sim}$ of X with $\varepsilon(f)\neq 0$ and $x\in X^{+}$, then $f.x=0$ implies $f(x)=0$.

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Introduction

A Riesz space E under an associative multiplication is said to be a Riesz algebra whenever the multiplication makes E an algebra (with the usual properties), and in addition it satisfies the following property : If $x, y\in E^{+}$, *then* $xy\in E^{+}$. A Riesz algebra E is said to be an $f-a\lg ebra$ whenever $x\wedge y=0$ implies $(xz)\wedge y=0$ for each $z\in E^{+}$. An order bounded band preserving operator is known as an orthomorphism and the set of all orthomorphism on X is denoted by $Orth(X)$, [2] .

A subset A of Riesz space is said to be bounded from above whenever there exists some x satisfying $y \le x$ for all $y \in A$. Similarly, a set A is said to be bounded from below whenever there exists some x such that $x \le y$ holds for all $y \in A$. Finally, a set A is called order bounded if it is bounded both from above and below . An operator $T : E \to F$ that maps order bounded subsets of E onto order bounded subsets of F is called order bounded. An operator $T : E \to E$ on a Riesz space is said to be band preserving whenever T leaves all bands of E invariant , i.e., whenever $T(B) \subseteq B$ holds for each band B of E.

Let E be a Riesz space. A linear functional $f : E \to R$ is called order bounded if f maps order bounded subsets of E onto bounded subsets of R , [4]. The vector space $E^{\sim}$ of all order bounded linear functionals on E is called the order dual of E , i.e., $E^{\sim} := L_b(E, R)$. Let X be an Archimedean $f-$ algebra with order dual $X^{\sim}$. Following construction $[3,5,6,7]$ a multiplication can be introduced in the order bidual $X^{\sim\sim}$ of X.
This is accomplished in four steps as explained below.

1) $Orth(X) \times X \to X$

$$(T, x) \to T.x = T(x) \qquad for \quad T \in Orth(X),\ x \in X .$$

2) $X \times X^{\sim} \to Orth(X)^{\sim}$

$$(x, x') \to (x.x')T = x'(Tx) \qquad for \quad x' \in X^{\sim},\ x \in X,\ T \in Orth(X).$$

3) $Orth(X)^{\sim\sim} \times X^{\sim} \to X^{\sim}$

$$(T, x') \to (T.x')x = T(x.x') \qquad for \ T \in Orth(X)^{\sim\sim}, x' \in X^{\sim},\ x \in X.$$

4) $Orth(X)^{\sim\sim} \times X^{\sim\sim} \to X^{\sim\sim}$

$$(T, \hat{x}) \to (T.\hat{x})(x') = \hat{x}(T.x') \qquad for \ T \in Orth(X)^{\sim\sim},\ x' \in X^{\sim\sim}.$$

Proposition 1 : Let $\alpha : Orth(X)^{\sim\sim} \to Orth(X^{\sim})$ be a mapping defined by $\alpha(T)x' = Tx' \ for \ T \in Orth(X)^{\sim\sim}$, $x' \in X^{\sim}$. Then α is a one-one , onto and algebra homomorphism.

Proof : We will prove the following, respectively.

i) α is a linear mapping.

ii) α is an one-one mapping.

iii) α is an algebra homomorphism.

i) $\alpha : Orth(X)^{\sim\sim} \to Orth(X^{\sim})$, $\alpha(T)x' = T.x'$, *for* $T \in Orth(X)^{\sim\sim}$, $x' \in X^{\sim}$.

$\alpha(T)x' = T.x'$

α is a linear mapping :

a) $\alpha(T+S) = \alpha(T) + \alpha(S)$ we must show that $\alpha(T+S)x' = \alpha(T)x' + \alpha(S)x'$ is true *for all* $x \in X$.

$[\alpha(T+S)x'].x = [(T+S).x'].x = (T+S)(x.x')$ From third product we have

$$
\begin{aligned}
&(T.x')x = T(x.x') \quad \text{and} \\
&(T+S)(x.x') = T(x.x') + S(x.x') \\
&\qquad = (T.x')x + (S.x')x \\
&\qquad = [\alpha(T)x'].x + [\alpha(S)x'].x.
\end{aligned}
$$

b) $\alpha(\lambda T) = \lambda\alpha(T)$ we must show that $\alpha(\lambda T)x' = \lambda\alpha(T)x'$ is true for all $x \in X$.

$$
\begin{aligned}
([\alpha(\lambda T).x']).x &= [(\lambda T).x'].x \quad , \quad x' \in X^{\sim} \\
&= \lambda T(x.x') \\
&= \lambda(T.x').x \\
&= \lambda[\alpha(T).x'].x. \qquad So, \alpha \ \ is \ a \ linear \ mapping.
\end{aligned}
$$

ii) α is one-one $[i.e.\ T \neq 0 \Rightarrow \alpha(T) \neq 0.]$

$0 \neq T \in Orth(X)^{\sim\sim}$ *there is a* $\exists\ y \in Orth(X)^{\sim}$

Let us take as $y = x.x' \in Orth(X)^{\sim}$

$Ty \neq 0$

$T(x.x') \neq 0$

$(T.x').x = 0$

$[\alpha(T)x']x \neq 0 \Rightarrow \alpha(T) \neq 0.$

iii) α is an algebra homomorphism, to prove this claim,

$$\alpha(T.S) = \alpha(T)\alpha(S) \qquad for\ all\ x \in X.$$
$$\begin{aligned}[\alpha(T.S).x'].x &= (T:S)(x.x') \quad , \quad x.x' \in Orth(X)^{\sim} \\ &= T(S(x.x')) \\ &= T(\alpha(S)x').x \\ &= T(S.x').x \quad , \quad S.x' \in X^{\sim} \\ &= [T(S.x')].x \\ &= \alpha(T)\alpha(S).\end{aligned}$$

Lemma 2[1,5] : Let $x \in X$, $f \in X^{\sim}$.
If the mapping $f.x : Orth(X) \to R$ *is defined by* $(f.x)(\pi) = (f \circ \pi).x$, *for* $\pi \in Orth(X)$ then $f.x \in OrthX^{\sim}$.

Proof : Consider the mapping (2)

$$X \times X^{\sim} \to OrthX^{\sim}$$
$$(x, f) \to x \circ f \quad , \quad x \circ f \in OrthX^{\sim} \quad , \quad x \circ f \in Orth(X) \to R \quad Then,\ we\ obtain$$
$$(x.f)(\pi) = (f \circ \pi).x.$$
Let $\beta : Orth(X) \to L_b(X)$ *be a mapping defined by*
$$T \to \beta(T)x = Tx.$$

Next, we obtain the following equalities:

$$\begin{aligned}(xf)(T) &= (Tx)(f) \quad , \quad f \in X^{\sim} \\ &= \beta(T)x\end{aligned}$$
$$\begin{aligned}(x.f)(\pi) &= (\pi x)(f) \\ &= (\beta(\pi)x).f \\ &= f(\beta(\pi)x) \\ &= f(\pi x). \qquad So,\end{aligned}$$
$$(x.f)(\pi) = f(\pi x) \quad and \quad xf \in OrthX^{\sim}.$$

Lemma 3 : Suppose the mapping $\phi : X^{\sim\sim} \to Orth(X)^{\sim\sim}$ is defined by

$\phi(F)(f) = F(f_x)$, *for* $F \in X$, $f \in Orth(X)^{\sim}$ *and the mapping*

$\alpha : Orth(X)^{\sim\sim} \to Orth(X^{\sim})$ *then*

$\beta = \alpha \circ \phi$ *is algebra homomorphism.*

$$
\begin{array}{lll}
 & \xrightarrow{\ \phi\ } & Orth(X)^{\sim\sim} \\
X^{\sim\sim} & & \downarrow \alpha \\
 & \xrightarrow{\ \beta = \alpha\circ\phi\ } & Orth(X^{\sim})
\end{array}
$$

Proof : We must show

$$\beta(F.G) = \beta(F).\beta(G) \quad for \quad F, G \in X^{\sim\sim}$$

$$
\begin{aligned}
\beta(F.G) &= (\alpha \circ \phi)(F.G) \\
&= \alpha(\phi(F.G)) \quad \textit{we know } \phi \textit{ is homomorphism}[1] \\
&= \alpha(\phi(F).\phi(G)) \\
&= \alpha(\phi(F).\alpha(\phi(G)) \quad (X^{\sim} \textit{ is } f-\text{module over } Orth(X)^{\sim\sim}) \\
&= (\alpha \circ \phi)(F)(\alpha \circ \phi)(G) \\
&= \beta(F)\beta(G).
\end{aligned}
$$

For every $f \in (X^{\sim})^{+}$, defined $\varepsilon(f)$ to be the set of all extensions of f in $\left(Orth(X)^{\sim}\right)^{+}$, [1]. That is , $\varepsilon(f) = \{g \in Orth(X)^{\sim} \ : \ 0 \le g \ and \ g_x = f \}$.

Proposition 4 : Let f be an element of order dual $X^{\sim}$ of X with $\varepsilon(f) \neq 0$. If $x \in X^{+}$, then $f.x = 0$ implies $f(x)=0$, [1].

Proof : Let us consider the set ,

$\varepsilon(f) = \{g \in Orth(X)^{\sim} \ : \ 0 \le g \ and \ g_x = f \}$. for $f \in X^{\sim}$, $x \in X$.

Let $0 = f.x \in Orth(X^{\sim})$

$0(T) = f(T.x) \quad , \quad \forall T \in Orth(X)$

$0 = f(Tx)$

If we take $T = I$

$f(Ix)=0 \;\Rightarrow\; f(x)=0$.

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