Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Bilinear multipliers of weighted Wiener amalgam spaces and variable exponent Wiener amalgam spaces

Öznur Kulak[1*] and Ahmet Turan Gürkanlı[2]

Dedicated to Professor Ravi P Agarwal.

*Correspondence:
oznurn@omu.edu.tr
[1]Department of Mathematics, Faculty of Arts and Sciences, Ondokuz Mayıs University, Kurupelit, Samsun, 55139, Turkey
Full list of author information is available at the end of the article

Abstract

Let ω_1, ω_2 be slowly increasing weight functions, and let ω_3 be any weight function on $\mathbb{R}^n$. Assume that $m(\xi,\eta)$ is a bounded, measurable function on $\mathbb{R}^n \times \mathbb{R}^n$. We define

$$B_m(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)m(\xi,\eta)e^{2\pi i\langle\xi+\eta,x\rangle}\,d\xi\,d\eta$$

for all $f,g \in C_c^\infty(\mathbb{R}^n)$. We say that $m(\xi,\eta)$ is a bilinear multiplier on $\mathbb{R}^n$ of type $(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))$ if B_m is a bounded operator from $W(L^{p_1},L^{q_1}_{\omega_1}) \times W(L^{p_2},L^{q_2}_{\omega_2})$ to $W(L^{p_3},L^{q_3}_{\omega_3})$, where $1 \le p_1 \le q_1 < \infty$, $1 \le p_2 \le q_2 < \infty$, $1 < p_3, q_3 \le \infty$. We denote by $\mathrm{BM}(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))$ the vector space of bilinear multipliers of type $(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))$. In the first section of this work, we investigate some properties of this space and we give some examples of these bilinear multipliers. In the second section, by using variable exponent Wiener amalgam spaces, we define the bilinear multipliers of type $(W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3))$ from $W(L^{p_1(x)},L^{q_1}_{\omega_1}) \times W(L^{p_2(x)},L^{q_2}_{\omega_2})$ to $W(L^{p_3(x)},L^{q_3}_{\omega_3})$, where $p_1^*,p_2^*,p_3^* < \infty$, $p_1(x) \le q_1$, $p_2(x) \le q_2$, $1 \le q_3 \le \infty$ for all $p_1(x),p_2(x),p_3(x) \in P(\mathbb{R}^n)$. We denote by $\mathrm{BM}(W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3))$ the vector space of bilinear multipliers of type $(W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3))$. Similarly, we discuss some properties of this space.

MSC: 42A45; 42B15; 42B35

Keywords: bilinear multipliers; weighted Wiener amalgam space; variable exponent Wiener amalgam space

1 Introduction

Throughout this paper we will work on $\mathbb{R}^n$ with Lebesgue measure dx. We denote by $C_c^\infty(\mathbb{R}^n)$, $C_c(\mathbb{R}^n)$ and $S(\mathbb{R}^n)$ the space of infinitely differentiable complex-valued functions with compact support on $\mathbb{R}^n$, the space of all continuous, complex-valued functions with compact support on $\mathbb{R}^n$ and the space of infinitely differentiable complex-valued functions on $\mathbb{R}^n$ that rapidly decrease at infinity, respectively. Let f be a complex-valued measurable function on $\mathbb{R}^n$. The translation, character and dilation operators T_x, M_x and D_s are defined by $T_xf(y) = f(y-x)$, $M_xf(y) = e^{2\pi i\langle x,y\rangle}f(y)$ and $D_t^pf(y) = t^{-\frac{n}{p}}f(\frac{y}{t})$, respectively, for $x, y \in \mathbb{R}^n$, $0 < p, t < \infty$. With this notation out of the way, one has, for $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$,

$$(T_x f)\hat{}(\xi) = M_{-x}\hat{f}(\xi), \qquad (M_x f)\hat{}(\xi) = T_x\hat{f}(\xi), \qquad \bigl(D_t^p f\bigr)\hat{}(\xi) = D_{t^{-1}}^{p'}\hat{f}(\xi).$$

For $1 \le p \le \infty$, $L^p(\mathbb{R}^n)$ denotes the usual Lebesgue space. A continuous function ω satisfying $1 \le \omega(x)$ and $\omega(x+y) \le \omega(x)\omega(y)$ for $x, y \in \mathbb{R}^n$ will be called a weight function on $\mathbb{R}^n$. If $\omega_1(x) \le \omega_2(x)$ for all $x \in \mathbb{R}^n$, we say that $\omega_1 \le \omega_2$. For $1 \le p \le \infty$, we set

$$L_\omega^p(\mathbb{R}^n) = \{f : f\omega \in L^p(\mathbb{R}^n)\}.$$

It is known that $L_\omega^p(\mathbb{R}^n)$ is a Banach space under the norm

$$\|f\|_{p,\omega} = \|f\omega\|_p = \left\{\int_{\mathbb{R}^n} \bigl|f(x)\omega(x)\bigr|^p \, dx\right\}^{\frac{1}{p}}, \quad 1 \le p < \infty$$

or

$$\|f\|_{\infty,\omega} = \|f\omega\|_\infty = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \bigl|f(x)\omega(x)\bigr|, \quad p = \infty.$$

The dual of the space $L_\omega^p(\mathbb{R}^n)$ is the space $L_{\omega^{-1}}^q(\mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $\omega^{-1}(x) = \frac{1}{\omega(x)}$. We say that a weight function υ_s is of polynomial type if $\upsilon_s(x) = (1 + |x|)^s$ for $s \ge 0$. Let f be a measurable function on $\mathbb{R}^n$. If there exist $C > 0$ and $N \in \mathbb{N}$ such that

$$\bigl|f(x)\bigr| \le C\bigl(1 + x^2\bigr)^N$$

for all $x \in \mathbb{R}^n$, then f is said to be a slowly increasing function [1]. It is easy to see that polynomial-type weight functions are slowly increasing. For $f \in L^1(\mathbb{R}^n)$, the Fourier transform of f is denoted by $\hat{f}$. We know that $\hat{f}$ is a continuous function on $\mathbb{R}^n$ which vanishes at infinity and it has the inequality $\|\hat{f}\|_\infty \le \|f\|_1$. We denote by $M(\mathbb{R}^n)$ the space of bounded regular Borel measures, by $M(\omega)$ the space of μ in $M(\mathbb{R}^n)$ such that

$$\|\mu\|_\omega = \int_{\mathbb{R}^n} \omega \, d|\mu| < \infty.$$

If $\mu \in M(\mathbb{R}^n)$, the Fourier-Stieltjes transform of μ is denoted by $\hat{\mu}$ [2].

The space $(L^p(\mathbb{R}^n))_{\text{loc}}$ consists of classes of measurable functions f on $\mathbb{R}^n$ such that $f\chi_K \in L^p(\mathbb{R}^n)$ for any compact subset $K \subset \mathbb{R}^n$, where χ_K is the characteristic function of K. Let us fix an open set $Q \subset \mathbb{R}^n$ with compact closure and $1 \le p, q \le \infty$. The weighted Wiener amalgam space $W(L^p, L_\omega^q)$ consists of all elements $f \in (L^p(\mathbb{R}^n))_{\text{loc}}$ such that $F_f(z) = \|f\chi_{z+Q}\|_p$ belongs to $L_\omega^q(\mathbb{R}^n)$; the norm of $W(L^p, L_\omega^q)$ is $\|f\|_{W(L^p, L_\omega^q)} = \|F_f\|_{q,\omega}$ [3–5].

In this paper, $P(\mathbb{R}^n)$ denotes the family of all measurable functions $p : \mathbb{R}^n \to [1, \infty)$. We put

$$p_* = \operatorname*{ess\,inf}_{x \in \mathbb{R}^n} p(x), \qquad p^* = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} p(x).$$

We shall also use the notation

$$\Omega_\infty = \{x \in \mathbb{R}^n : p(x) = \infty\}.$$

The variable exponent Lebesgue spaces (or generalized Lebesgue spaces) $L^{p(x)}(\mathbb{R}^n)$ are defined as the set of all (equivalence classes) measurable functions f on $\mathbb{R}^n$ such that $\varrho_p(\lambda f) < \infty$ for some $\lambda > 0$, equipped with the Luxemburg norm

$$\|f\|_{p(x)} = \inf\left\{\lambda > 0 : \varrho_p\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

If $p^* < \infty$, then $f \in L^{p(x)}(\mathbb{R}^n)$ if $\varrho_p(f) < \infty$. The set $L^{p(x)}(\mathbb{R}^n)$ is a Banach space with the norm $\|\cdot\|_{p(x)}$. If $p(x) = p$ is a constant function, then the norm $\|\cdot\|_{p(x)}$ coincides with the usual Lebesgue norm $\|\cdot\|_p$ [6]. The spaces $L^{p(x)}(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ have many common properties. A crucial difference between $L^{p(x)}(\mathbb{R}^n)$ and the classical Lebesgue spaces $L^p(\mathbb{R}^n)$ is that $L^{p(x)}(\mathbb{R}^n)$ is not invariant under translation in general. If $p^* < \infty$, then $C_c^\infty(\mathbb{R}^n)$ is dense in $L^{p(x)}(\mathbb{R}^n)$. The space $L^{p(x)}(\mathbb{R}^n)$ is a solid space, that is, if $f \in L^{p(x)}(\mathbb{R}^n)$ is given and $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies $|g(x)| \le |f(x)|$ a.e., then $g \in L^{p(x)}(\mathbb{R}^n)$ and $\|g\|_{p(x)} \le \|f\|_{p(x)}$ by [6]. In this paper we will assume that $p^* < \infty$.

The space $(L^{p(x)}(\mathbb{R}^n))_{\mathrm{loc}}$ consists of classes of measurable functions f on $\mathbb{R}^n$ such that $f\chi_K \in L^{p(x)}(\mathbb{R}^n)$ for any compact subset $K \subset \mathbb{R}^n$. Let us fix an open set $Q \subset \mathbb{R}^n$ with compact closure, $p(x) \in P(\mathbb{R}^n)$ and $1 \le q \le \infty$. The variable exponent amalgam space $W(L^{p(x)}, L^q_\omega)$ consists of all elements $f \in (L^{p(x)}(\mathbb{R}^n))_{\mathrm{loc}}$ such that $F_f(z) = \|f\chi_{z+Q}\|_{p(x)}$ belongs to $L^q_\omega(\mathbb{R}^n)$; the norm of $W(L^{p(x)}, L^q_\omega)$ is $\|f\|_{W(L^{p(x)}, L^q_\omega)} = \|F_f\|_{q,\omega}$ [7].

2 The bilinear multipliers space BM[$W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)$]

Lemma 2.1 *Let $1 \le p \le q < \infty$ and ω be a slowly increasing weight function. Then $C_c^\infty(\mathbb{R}^n)$ is dense in the Wiener amalgam space $W(L^p, L^q_\omega)$.*

Proof Since $\overline{C_c(\mathbb{R}^n)} = L^q_\omega(\mathbb{R}^n)$ [8], we have $\overline{C_c(\mathbb{R}^n)} = W(L^p, L^q_\omega)$ by a lemma in [9]. Also we have the inclusion

$$C_c^\infty(\mathbb{R}^n) \subset C_c(\mathbb{R}^n) \subset W(L^p, L^q_\omega).$$

For the proof that $C_c^\infty(\mathbb{R}^n)$ is dense in $W(L^p, L^q_\omega)$, take any $f \in W(L^p, L^q_\omega)$. For given $\varepsilon > 0$, there exists $g \in C_c(\mathbb{R}^n)$ such that

$$\|f - g\|_{W(L^p, L^q_\omega)} < \frac{\varepsilon}{2}. \tag{2.1}$$

Also, since $g \in C_c(\mathbb{R}^n) \subset L^q_\omega(\mathbb{R}^n)$ and $C_c^\infty(\mathbb{R}^n)$ is dense in $L^q_\omega(\mathbb{R}^n)$, by Lemma 2.1 in [10], there exists $h \in C_c^\infty(\mathbb{R}^n)$ such that

$$\|g - h\|_{q,\omega} < \frac{\varepsilon}{2}.$$

Furthermore, by using the inequality $p \le q$, we write

$$\|g - h\|_{W(L^p, L^q_\omega)} \le \|g - h\|_{q,\omega} < \frac{\varepsilon}{2} \tag{2.2}$$

(see [11] and [5]). Combining (2.1) and (2.2), we obtain

$$\|f-h\|_{W(L^p,L^q_\omega)} \le \|f-g\|_{W(L^p,L^q_\omega)} + \|h-g\|_{W(L^p,L^q_\omega)} < \varepsilon.$$

This completes the proof. □

Definition 2.1 Let $1 \le p_1 \le q_1 < \infty$, $1 \le p_2 \le q_2 < \infty$, $1 < p_3, q_3 \le \infty$ and ω_1, ω_2, ω_3 be weight functions on $\mathbb{R}^n$. Assume that ω_1, ω_2 are slowly increasing functions and $m(\xi,\eta)$ is a bounded, measurable function on $\mathbb{R}^n \times \mathbb{R}^n$. Define

$$B_m(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)m(\xi,\eta)e^{2\pi i\langle \xi+\eta,x\rangle}\,d\xi\,d\eta$$

for all $f,g \in C_c^\infty(\mathbb{R}^n)$.

m is said to be a bilinear multiplier on $\mathbb{R}^n$ of type $(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))$ if there exists $C > 0$ such that

$$\|B_m(f,g)\|_{W(L^{p_3},L^{q_3}_{\omega_3})} \le C\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega 2})}$$

for all $f,g \in C_c^\infty(\mathbb{R}^n)$. That means that B_m extends to a bounded bilinear operator from $W(L^{p_1},L^{q_1}_{\omega_1}) \times W(L^{p_2},L^{q_2}_{\omega 2})$ to $W(L^{p_3},L^{q_3}_{\omega_3})$.

We denote by $\mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$ the space of all bilinear multipliers of type $(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))$ and $\|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))} = \|B_m\|$.

The following theorem is an example to a bilinear multiplier on $\mathbb{R}^n$ of type $(W(p_1,q_1,\omega_1;$ $p_2,q_2,\omega_2;p_3,q_3,\omega_3))$.

Theorem 2.1 *Let $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}$, $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q_3}$ and $\omega_3 \le \omega_1$. If $K \in L^1_{\omega_3}(\mathbb{R}^n)$, then $m(\xi,\eta) = \hat{K}(\xi-\eta)$ defines a bilinear multiplier and $\|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))} \le \|K\|_{1,\omega_3}$.*

Proof We know by Theorem 2.1 in [10] that for $f,g \in C_c^\infty(\mathbb{R}^n)$,

$$B_m(f,g)(t) = \int_{\mathbb{R}^n} f(t-y)g(t+y)K(y)\,dy. \tag{2.3}$$

Also by Proposition 11.4.1 in [5], $T_yf \in W(L^{p_1},L^{q_1}_{\omega_1})$, $T_{-y}g \in W(L^{p_2},L^{q_2}_{\omega 2})$. So, we write $F_{T_yf} \in L^{q_1}_{\omega_1}(\mathbb{R}^n)$, $F_{T_{-y}g} \in L^{q_2}_{\omega 2}(\mathbb{R}^n)$.

Using the Minkowski inequality and the generalized Hölder inequality, we have

$$\begin{aligned}
\|B_m(f,g)\|_{W(L^{p_3},L^{q_3}_{\omega_3})} &= \big\|\|B_m(f,g)\chi_{Q+x}\|_{p_3}\big\|_{q_3,\omega_3} \\
&= \left\|\left\|\left\{\int_{\mathbb{R}^n} f(t-y)g(t+y)K(y)\,dy\right\}\chi_{Q+x}\right\|_{p_3}\right\|_{q_3,\omega_3} \\
&\le \int_{\mathbb{R}^n}\big\|\|f(t-y)g(t+y)\chi_{Q+x}(t)\|_{p_3}\big\|_{q_3,\omega_3}|K(y)|\,dy \\
&\le \int_{\mathbb{R}^n}\big\|\|f(t-y)\chi_{Q+x}(t)\|_{p_1}\|g(t+y)\chi_{Q+x}(t)\|_{p_2}\big\|_{q_3,\omega_3}|K(y)|\,dy \\
&= \int_{\mathbb{R}^n}\|F_{T_yf}(x)\omega_3(x)F_{T_{-y}g}(x)\|_{q_3}|K(y)|\,dy.
\end{aligned} \tag{2.4}$$

Again, by using Proposition 11.4.1 in [5] and the assumption $\omega_3 \le \omega_1$, we write

$$\|F_{T_y f}\omega_3\|_{q_1} \le \omega_3(y)\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})} < \infty. \tag{2.5}$$

From this result, we find $F_{T_y f} \in L^{q_1}_{\omega_3}(\mathbb{R}^n)$. Hence by (2.4), (2.5) and the generalized Hölder inequality, we obtain

$$\begin{aligned}\|B_m(f,g)\|_{W(L^{p_3},L^{q_3}_{\omega_3})} &\le \int_{\mathbb{R}^n} \|F_{T_y f}(x)\omega_3(x)\|_{q_1}\|F_{T_{-y}g}(x)\|_{q_2}|K(y)|\,dy \\ &\le \int_{\mathbb{R}^n} \|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}|K(y)|\omega_3(y)\,dy \\ &= \|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}\|K\|_{1,\omega_3} \\ &= C\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})},\end{aligned} \tag{2.6}$$

where $C = \|K\|_{1,\omega_3}$. Then $m(\xi,\eta) = \hat{K}(\xi-\eta)$ defines a bilinear multiplier. Finally, using (2.6), we obtain

$$\begin{aligned}\|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))} &= \sup_{\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\le 1,\|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}\le 1} \frac{\|B_m(f,g)\|_{W(L^{p_3},L^{q_3}_{\omega_3})}}{\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}} \\ &\le \|K\|_{1,\omega_3}.\end{aligned}$$

□

Definition 2.2 Let $1 \le p_1 \le p_2 < \infty$, $1 \le q_1 \le q_2 < \infty$, $1 < p_3, q_3 \le \infty$ and ω_1, ω_2, ω_3 be weight functions on $\mathbb{R}^n$. Suppose that ω_1, ω_2 are slowly increasing functions. We denote by $\tilde{M}[(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$ the space of measurable functions $M : \mathbb{R}^n \to \mathbb{C}$ such that $m(\xi,\eta) = M(\xi-\eta) \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$, that is to say,

$$B_M(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)M(\xi-\eta)e^{2\pi i\langle\xi+\eta,x\rangle}\,d\xi\,d\eta$$

extends to a bounded bilinear map from $W(L^{p_1},L^{q_1}_{\omega_1}) \times W(L^{p_2},L^{q_2}_{\omega_2})$ to $W(L^{p_3},L^{q_3}_{\omega_3})$. We denote $\|M\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))} = \|B_M\|$.

Let ω be a weight function. The continuous function ω^{-1} cannot be a weight function. But the following lemma can be proved easily by using the technique of the proof of Lemma 2.1.

Lemma 2.2 *Let $1 \le p \le q < \infty$ and ω be a slowly increasing continuous weight function. Then $C_c^\infty(\mathbb{R}^n)$ is dense in $W(L^p,L^q_{\omega^{-1}})$ Wiener amalgam space.*

Theorem 2.2 *Let $\frac{1}{p_3}+\frac{1}{p_3'}=1$, $\frac{1}{q_3}+\frac{1}{q_3'}=1$, $q_3' \ge p_3' \ge 1$ and ω_3 be a continuous, symmetric slowly increasing weight function. Then $m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$ if and only if there exists $C > 0$ such that*

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)\,d\xi\,d\eta\right| \le C\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}\|h\|_{W(L^{p_3'},L^{q_3'}_{\omega^{-1}})}$$

for all $f,g,h \in C_c^\infty(\mathbb{R}^n)$.

Proof We assume that $m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$. By Theorem 2.2 in [10], we write, for all $f,g,h \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)\,d\xi\,d\eta\right| &= \left|\int_{\mathbb{R}^n} h(y)\tilde{B}_m(f,g)(y)\,dy\right| \\ &\leq \int_{\mathbb{R}^n}\left|h(y)\right|\left|\tilde{B}_m(f,g)(y)\right|dy, \end{aligned}\tag{2.7}$$

where $\tilde{B}_m(f,g)(y) = B_m(f,g)(-y)$. If we set $-t = u$, we have

$$\begin{aligned}\left\|\tilde{B}_m(f,g)\right\|_{W(L^{p_3},L^{q_3}_{\omega_3})} &= \left\|F^Q_{\tilde{B}_m(f,g)}\right\|_{q_3,\omega_3} = \left\|\left\|B_m(f,g)(u)\chi_{Q+x}(-u)\right\|_{p_3}\right\|_{q_3,\omega_3} \\ &= \left\|\left\|B_m(f,g)(u)\chi_{-Q-x}(u)\right\|_{p_3}\right\|_{q_3,\omega_3} = \left\|F^{-Q}_{B_m(f,g)}(-x)\right\|_{q_3,\omega_3}.\end{aligned}\tag{2.8}$$

Since ω_3 is a symmetric weight function, if we set $-x = y$, we have

$$\left\|\tilde{B}_m(f,g)\right\|_{W(L^{p_3},L^{q_3}_{\omega_3})} = \left\|F^{-Q}_{B_m(f,g)}(y)\right\|_{q_3,\omega_3}.\tag{2.9}$$

We know from [3] and [5] that the definition of $W(L^{p_3},L^{q_3}_{\omega_3})$ is independent of the choice of Q. Then there exists $C > 0$ such that

$$\left\|F^{-Q}_{B_m(f,g)}(y)\right\|_{q_3,\omega_3} \leq C_1\left\|F^{Q}_{B_m(f,g)}(y)\right\|_{q_3,\omega_3}.\tag{2.10}$$

Hence, by (2.9) and (2.10), we have

$$\left\|\tilde{B}_m(f,g)\right\|_{W(L^{p_3},L^{q_3}_{\omega_3})} \leq C_1\left\|F^{Q}_{B_m(f,g)}(y)\right\|_{q_3,\omega_3} = C_1\left\|B_m(f,g)\right\|_{W(L^{p_3},L^{q_3}_{\omega_3})}.\tag{2.11}$$

Since from the assumption $m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$ the right-hand side of (2.11) is finite, thus $\tilde{B}_m(f,g) \in W(L^{p_3},L^{q_3}_{\omega_3})$. On the other hand, since $m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$, there exists $C_2 > 0$ such that

$$\left\|B_m(f,g)\right\|_{W(L^{p_3},L^{q_3}_{\omega_3})} \leq C_2\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}.\tag{2.12}$$

Combining (2.11) and (2.12), we have

$$\left\|\tilde{B}_m(f,g)\right\|_{W(L^{p_3},L^{q_3}_{\omega_3})} \leq C_1C_2\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}.\tag{2.13}$$

If we apply the Hölder inequality to the right-hand side of inequality (2.7) and use inequality (2.13), we obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)\,d\xi\,d\eta\right| \\ &\quad\leq \left\|\tilde{B}_m(f,g)\right\|_{W(L^{p_3},L^{q_3}_{\omega_3})}\|h\|_{W(L^{p_3'},L^{q_3'}_{\omega^{-1}})} \\ &\quad\leq C_1C_2\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}\|h\|_{W(L^{p_3'},L^{q_3'}_{\omega_3^{-1}})}.\end{aligned}$$

For the proof of converse, assume that there exists a constant $C > 0$ such that

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)\,d\xi\,d\eta\right|\\&\quad\leq C\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}\|h\|_{W(L^{p_3'},L^{q_3'}_{\omega_3^{-1}})}\end{aligned}\tag{2.14}$$

for all $f,g,h\in C_c^\infty(\mathbb{R}^n)$. From the assumption and (2.14), we write

$$\left|\int_{\mathbb{R}^n}h(y)\tilde{B}_m(f,g)(y)\,dy\right|\leq C\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}\|h\|_{W(L^{p_3'},L^{q_3'}_{\omega_3^{-1}})}.\tag{2.15}$$

Define a function ℓ from $C_c^\infty(\mathbb{R}^n)\subset W(L^{p_3'},L^{q_3'}_{\omega^{-1}})$ to $\mathbb{C}$ such that

$$\ell(h)=\int_{\mathbb{R}^n}h(y)\tilde{B}_m(f,g)(y)\,dy.$$

ℓ is linear and bounded by (2.15). Also, since $q_3'\geq p_3'\geq 1$, we have $\overline{C_c^\infty(\mathbb{R}^n)}=W(L^{p_3'},L^{q_3'}_{\omega_3^{-1}})$ by Lemma 2.2. Thus ℓ extends to a bounded function from $W(L^{p_3'},L^{q_3'}_{\omega_3^{-1}})$ to $\mathbb{C}$. Then $\ell\in(W(L^{p_3'},L^{q_3'}_{\omega_3^{-1}}))^*=W(L^{p_3},L^{q_3}_{\omega_3})$. Again, since the definition of $W(L^{p_3},L^{q_3}_{\omega_3})$ is independent of the choice of Q, there exists $C_3>0$ such that

$$\left\|B_m(f,g)\right\|_{W(L^{p_3},L^{q_3}_{\omega_3})}\leq C_3\left\|\tilde{B}_m(f,g)\right\|_{W(L^{p_3},L^{q_3}_{\omega_3})}.\tag{2.16}$$

Combining (2.15) and (2.16), we obtain

$$\begin{aligned}\left\|B_m(f,g)\right\|_{W(L^{p_3},L^{q_3}_{\omega_3})}&\leq C_3\left\|\tilde{B}_m(f,g)\right\|_{W(L^{p_3},L^{q_3}_{\omega_3})}\\&=C_3\|\ell\|=C_3\sup_{\|h\|_{W(L^{p_3'},L^{q_3'}_{\omega_3^{-1}})}\leq 1}\frac{|\ell(h)|}{\|h\|_{W(L^{p_3'},L^{q_3'}_{\omega_3^{-1}})}}\\&\leq C_3C\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}.\end{aligned}$$

This completes proof. □

The following theorem is a generalization of Theorem 2.1.

Theorem 2.3 *Let* $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}$, $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q_3}$, $\omega_3\leq\omega_1$ *and* $\upsilon(x)=C(1+x^2)^N$, $C\geq 0$, $N\in\mathbb{N}$ *be a weight function. If* $\mu\in M(\upsilon)$ *and* $m(\xi,\eta)=\hat{\mu}(\alpha\xi+\beta\eta)$ *for* $\alpha,\beta\in\mathbb{R}$, *then* $m\in\mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$. *Moreover,*

$$\begin{aligned}&\|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))}\leq\|\mu\|_\upsilon\quad\text{if }|\alpha|\leq 1,\\&\|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))}\leq|\alpha|^{2N}\|\mu\|_\upsilon\quad\text{if }|\alpha|>1.\end{aligned}$$

Proof Let $f,g\in C_c^\infty(\mathbb{R}^n)$. By Theorem 2.3 in [10], we have

$$B_m(f,g)(t)=\int_{\mathbb{R}^n}f(t-\alpha y)g(t-\beta y)\,d\mu(y).\tag{2.17}$$

Also by [5] we write the inequalities

$$\|T_{\alpha y}f\|_{W(L^{p_1},L^{q_1}_{\omega_1})} \le \omega_1(\alpha y)\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})} \tag{2.18}$$

and

$$\|T_{\beta y}g\|_{W(L^{p_2},L^{q_2}_{\omega 2})} \le \omega_2(\alpha y)\|g\|_{W(L^{p_2},L^{q_2}_{\omega 2})}. \tag{2.19}$$

From these inequalities, we have $F_{T_{\alpha y}f} \in L^{q_1}_{\omega_1}(\mathbb{R}^n)$ and $F_{T_{\beta y}g} \in L^{q_2}_{\omega 2}(\mathbb{R}^n)$. If we use the inequality $\omega_3 \le \omega_1$ and set $x - \alpha t = u$, we obtain

$$\|F_{T_{\alpha y}f}\omega_3\|_{q_1} \le \|F_{T_{\alpha y}f}\omega_1\|_{q_1} \le \omega_1(\alpha y)\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}, \tag{2.20}$$

and hence $F_{T_{\alpha y}f}\omega_3 \in L^{q_1}(\mathbb{R}^n)$. Then by (2.17), (2.18), (2.19), (2.20) and the Hölder inequality, we have

$$\begin{aligned}
\|B_m(f,g)\|_{W(L^{p_3},L^{q_3}_{\omega_3})} &\le \left\| \int_{\mathbb{R}^n} \|f(t-\alpha y)g(t-\beta y)\chi_{Q+x}(t)\|_{p_3}\, d|\mu|(y) \right\|_{q_3,\omega_3} \\
&\le \left\| \int_{\mathbb{R}^n} \|f(t-\alpha y)\chi_{Q+x}(t)\|_{p_1} \|g(t-\beta y)\chi_{Q+x}(t)\|_{p_2}\, d|\mu|(y) \right\|_{q_3,\omega_3} \\
&\le \int_{\mathbb{R}^n} \|F_{T_{\alpha y}f}(x)F_{T_{\beta y}g}(x)\|_{q_3,\omega_3}\, d|\mu|(y) \\
&\le \int_{\mathbb{R}^n} \|F_{T_{\alpha y}f}(x)\omega_3(x)\|_{q_1} \|F_{T_{\beta y}g}(x)\|_{q_2}\, d|\mu|(y) \\
&\le \int_{\mathbb{R}^n} \omega_1(\alpha y)\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2})}\, d|\mu|(y) \\
&\le \|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega 2})} \int_{\mathbb{R}^n} \omega_1(\alpha y)\, d|\mu|(y).
\end{aligned} \tag{2.21}$$

Now, suppose that $\alpha \le 1$. Since ω_1 is a slowly increasing weight function, there exist $C \ge 0$ and $N \in \mathbb{N}$ such that

$$\omega_1(x) \le C\left(1+x^2\right)^N = \upsilon(x).$$

Then

$$\int_{\mathbb{R}^n} \omega_1(\alpha y)\, d|\mu|(y) \le \int_{\mathbb{R}^n} C\left(1+\alpha^2 y^2\right)^N d|\mu|(y) \le \int_{\mathbb{R}^n} C\left(1+y^2\right)^N d|\mu|(y) = \|\mu\|_\upsilon.$$

Hence by (2.21)

$$\|B_m(f,g)\|_{W(L^{p_3},L^{q_3}_{\omega_3})} \le \|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega 2})}\|\mu\|_\upsilon. \tag{2.22}$$

Thus $m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$, and by (2.22) we obtain

$$\begin{aligned}
\|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))} &= \sup_{\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\le 1, \|g\|_{W(L^{p_2},L^{q_2}_{\omega 2})}\le 1} \frac{\|B_m(f,g)\|_{W(L^{p_3},L^{q_3}_{\omega_3})}}{\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega 2})}} \\
&\le \|\mu\|_\upsilon.
\end{aligned}$$

Similarly, if $\alpha > 1$, then

$$\int_{\mathbb{R}^n} \omega_1(\alpha y)\, d|\mu|(y) \le \int_{\mathbb{R}^n} C\bigl(\alpha^2 + \alpha^2 y^2\bigr)^N d|\mu|(y) = \alpha^{2N} \int_{\mathbb{R}^n} \upsilon(y)\, d|\mu|(y) = \alpha^{2N} \|\mu\|_\upsilon.$$

Therefore by (2.21) we have

$$\bigl\|B_m(f,g)\bigr\|_{W(L^{p_3},L^{q_3}_{\omega_3})} \le \alpha^{2N} \|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})} \|\mu\|_\upsilon. \tag{2.23}$$

Hence, we obtain $m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$, and by (2.23)

$$\begin{aligned} \|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))} &= \sup_{\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})} \le 1, \|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})} \le 1} \frac{\|B_m(f,g)\|_{W(L^{p_3},L^{q_3}_{\omega_3})}}{\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}} \\ &\le \alpha^{2N} \|\mu\|_\upsilon. \end{aligned}$$

□

Now, we will give some properties of the space $\mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$.

Theorem 2.4 *Let $m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$.*

(a) *$T_{(\xi_0,\eta_0)} m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$ for each $(\xi_0,\eta_0) \in \mathbb{R}^{2n}$ and*

$$\|T_{(\xi_0,\eta_0)} m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))} = \|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))}.$$

(b) *$M_{(\xi_0,\eta_0)} m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$ for each $(\xi_0,\eta_0) \in \mathbb{R}^{2n}$ and*

$$\begin{aligned} &\|M_{(\xi_0,\eta_0)} m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))} \\ &\quad \le \omega_1(-\xi_0)\omega_2(-\eta_0) \|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))}. \end{aligned}$$

Proof (a) Let $f,g \in C_c^\infty(\mathbb{R}^n)$. From Theorem 2.4 in [10], we write the equality

$$B_{T_{(\xi_0,\eta_0)} m}(f,g)(x) = e^{2\pi i \langle \xi_0 + \eta_0, x\rangle} B_m(M_{-\xi_0} f, M_{-\eta_0} g)(x). \tag{2.24}$$

Also the equalities $\|M_{-\xi_0} f\|_{W(L^{p_1},L^{q_1}_{\omega_1})} = \|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}$ and $\|M_{-\eta_0} g\|_{W(L^{p_2},L^{q_2}_{\omega_2})} = \|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}$ are satisfied. Then, using equality (2.24) and the assumption $m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$, we have

$$\begin{aligned} \bigl\|B_{T_{(\xi_0,\eta_0)} m}(f,g)\bigr\|_{W(L^{p_3},L^{q_3}_{\omega_3})} &= \bigl\|e^{2\pi i \langle \xi_0 + \eta_0, x\rangle} B_m(M_{-\xi_0} f, M_{-\eta_0} g)\bigr\|_{W(L^{p_3},L^{q_3}_{\omega_3})} \\ &= \bigl\|B_m(M_{-\xi_0} f, M_{-\eta_0} g)\bigr\|_{W(L^{p_3},L^{q_3}_{\omega_3})} \\ &\le C\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})} \end{aligned}$$

for some $C > 0$. Thus $T_{(\xi_0,\eta_0)} m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$. Moreover, we obtain

$$\begin{aligned} &\|T_{(\xi_0,\eta_0)} m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))} \\ &\quad = \|B_{T_{(\xi_0,\eta_0)} m}\| = \sup_{\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})} \le 1, \|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})} \le 1} \frac{\|B_{T_{(\xi_0,\eta_0)} m}(f,g)\|_{W(L^{p_3},L^{q_3}_{\omega_3})}}{\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}} \end{aligned}$$

$$= \sup_{\|M_{-\xi_0}f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\le 1, \|M_{-\eta_0}g\|_{W(L^{p_2},L^{q_2}_{\omega 2})}\le 1} \frac{\|B_{T_{(\xi_0,\eta_0)}m}(M_{-\xi_0}f, M_{-\eta_0}g)\|_{W(L^{p_3},L^{q_3}_{\omega_3})}}{\|M_{-\xi_0}f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|M_{-\eta_0}g\|_{W(L^{p_2},L^{q_2}_{\omega 2})}}$$

$$= \|B_m\| = \|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))}.$$

(b) For any $f, g \in C_c^\infty(\mathbb{R}^n)$, we write

$$B_{M_{(\xi_0,\eta_0)}m}(f,g)(x) = B_m(T_{-\xi_0}f, T_{-\eta_0}g)(x) \tag{2.25}$$

by Theorem 2.4 in [10]. Also, the inequalities $\|T_{-\xi_0}f\|_{W(L^{p_1},L^{q_1}_{\omega_1})} \le \omega_1(-\xi_0)\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}$ and $\|T_{-\eta_0}g\|_{W(L^{p_2},L^{q_2}_{\omega 2})} \le \omega_2(-\eta_0)\|g\|_{W(L^{p_2},L^{q_2}_{\omega 2})}$ are satisfied [5]. Since $m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$, by (2.25) we have

$$\begin{aligned}
\|B_{M_{(\xi_0,\eta_0)}m}(f,g)\|_{W(L^{p_3},L^{q_3}_{\omega_3})} &= \|B_m(T_{-\xi_0}f, T_{-\eta_0}g)\|_{W(L^{p_3},L^{q_3}_{\omega_3})} \\
&\le \|B_m\|\|T_{-\xi_0}f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|T_{-\eta_0}g\|_{W(L^{p_2},L^{q_2}_{\omega 2})} \\
&\le \omega_1(-\xi_0)\omega_2(-\eta_0)\|B_m\|\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega 2})}
\end{aligned} \tag{2.26}$$

and hence $M_{(\xi_0,\eta_0)}m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$. So, by (2.26) we obtain

$$\begin{aligned}
&\|M_{(\xi_0,\eta_0)}m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))} \\
&\quad = \sup_{\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\le 1, \|g\|_{W(L^{p_2},L^{q_2}_{\omega 2})}\le 1} \frac{\|B_{M_{(\xi_0,\eta_0)}m}(f,g)\|_{W(L^{p_3},L^{q_3}_{\omega_3})}}{\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega 2})}} \\
&\quad \le \omega_1(-\xi_0)\omega_2(-\eta_0)\|B_m\| = \omega_1(-\xi_0)\omega_2(-\eta_0)\|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))}. \qquad \square
\end{aligned}$$

Lemma 2.3 *If ω is a slowly increasing weight function such that $\omega(x) \le C_1(1+x^2)^N = \upsilon(x)$ and $f \in W(L^p, L^q_\omega)$, then $D^p_y f \in W(L^p, L^q_\upsilon)$. Moreover,*

$$\begin{aligned}
&\|D^p_y f\|_{W(L^p,L^q_\omega)} \le C\|f\|_{W(L^p,L^q_\upsilon)} \quad \text{if } y \le 1, \\
&\|D^p_y f\|_{W(L^p,L^q_\omega)} < C y^{\frac{n}{q}+2N}\|f\|_{W(L^p,L^q_\upsilon)} \quad \text{if } y > 1
\end{aligned}$$

for some $C > 0$.

Proof Take any $f \in W(L^p, L^q_\omega)$. If we get $\frac{t}{y} = u$, we obtain

$$\begin{aligned}
\|D^p_y f\|_{W(L^p,L^q_\omega)} &= \left\| \left\{ \int_{\mathbb{R}^n} |D^p_y f(t)\chi_{Q+x}(t)|^p \, dt \right\}^{\frac{1}{p}} \right\|_{q,\omega} \\
&= \left\| \left\{ \int_{\mathbb{R}^n} \left| y^{-\frac{n}{p}} f\left(\frac{t}{y}\right)\right|^p \chi_{Q+x}(t) \, dt \right\}^{\frac{1}{p}} \right\|_{q,\omega} \\
&= \left\| \left\{ \int_{\mathbb{R}^n} |f(u)|^p \chi_{y^{-1}Q+y^{-1}x}(u) \, du \right\}^{\frac{1}{p}} \right\|_{q,\omega} \\
&= \|F_f^{y^{-1}Q}(y^{-1}x)\|_{q,\omega}.
\end{aligned} \tag{2.27}$$

Again, if we say $y^{-1}x = s$ and use ω to be slowly increasing, then there exist $C_1 > 0$ and $N \in \mathbb{N}$ such that

$$\begin{aligned}\|D_y^p f\|_{W(L^p,L^q_\omega)} &= \left\{\int_{\mathbb{R}^n} \left|F_f^{y^{-1}Q}(y^{-1}x)\right|^q \omega(x)^q\,dx\right\}^{\frac{1}{q}}\\ &= y^{\frac{n}{q}}\left\{\int_{\mathbb{R}^n} \left|F_f^{y^{-1}Q}(s)\right|^q \omega(ys)^q\,ds\right\}^{\frac{1}{q}}\\ &\leq y^{\frac{n}{q}}\left\{\int_{\mathbb{R}^n} \left|F_f^{y^{-1}Q}(s)\right|^q \left(C_1\left(1+y^2s^2\right)^N\right)^q\,ds\right\}^{\frac{1}{q}} \qquad (2.28)\end{aligned}$$

by equation (2.27).

Let $y \leq 1$. Using inequality (2.28), we have

$$\|D_y^p f\|_{W(L^p,L^q_\omega)} \leq \left\{\int_{\mathbb{R}^n} \left|F_f^{y^{-1}Q}(s)\right|^q \left(C_1\left(1+s^2\right)^N\right)^q\,ds\right\}^{\frac{1}{q}} = \left\|F_f^{y^{-1}Q}\right\|_{q,\upsilon}. \qquad (2.29)$$

Since $y^{-1}Q$ is a compact set and the definition of $W(L^p,L^q_\upsilon)$ is independent of the choice of a compact set Q, then there exists $C > 0$ such that

$$\left\|F_f^{y^{-1}Q}\right\|_{q,\upsilon} \leq C\left\|F_f^{Q}\right\|_{q,\upsilon} \qquad (2.30)$$

by [3, 5]. Then by (2.29) we write

$$\|D_y^p f\|_{W(L^p,L^q_\omega)} \leq \left\|F_f^{y^{-1}Q}\right\|_{q,\upsilon} \leq C\left\|F_f^{Q}\right\|_{q,\upsilon} = C\|f\|_{W(L^p,L^q_\upsilon)}.$$

Thus we have $D_y^p f \in W(L^p,L^q_\upsilon)$.

Now, assume that $y > 1$. Similarly, by (2.28) and (2.30), we get

$$\begin{aligned}\|D_y^p f\|_{W(L^p,L^q_\omega)} &< y^{\frac{n}{q}}\left\{\int_{\mathbb{R}^n} \left|F_f^{y^{-1}Q}(s)\right|^q \left(C_1\left(y^2+y^2s^2\right)^N\right)^q\,ds\right\}^{\frac{1}{q}}\\ &= y^{\frac{n}{q}+2N}\left\{\int_{\mathbb{R}^n} \left|F_f^{y^{-1}Q}(s)\right|^q \upsilon(s)^q\,ds\right\}^{\frac{1}{q}}\\ &\leq y^{\frac{n}{q}+2N}\|f\|_{W(L^p,L^q_\upsilon)}.\end{aligned}$$

Hence $D_y^p f \in W(L^p,L^q_\upsilon)$. □

Theorem 2.5 *Let $\upsilon_i(x) = C_i(1+x^2)^{N_i}$, $C_i > 0$, $N_i > 0$ for $i = 1,2,3$, and let ω_3 be a slowly increasing weight function. If $\frac{2}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3}$, $0 < y < \infty$ and $m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\upsilon_3)]$, then $D_y^q m \in \mathrm{BM}[W(p_1,q_1,\upsilon_1;p_2,q_2,\upsilon_2;p_3,q_3,\omega_3)]$. Moreover, then*

$$\begin{aligned}&\|D_y^q m\|_{(W(p_1,q_1,\upsilon_1;p_2,q_2,\upsilon_2;p_3,q_3,\omega_3))}\\ &\quad\leq Cy^{-\frac{n}{q_3}-2N_3}\|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\upsilon_3))} \quad \text{if } y \leq 1,\\ &\|D_y^q m\|_{(W(p_1,q_1,\upsilon_1;p_2,q_2,\upsilon_2;p_3,q_3,\omega_3))}\\ &\quad< Cy^{\frac{n}{q_1}+\frac{n}{q_2}+2N_1+2N_2}\|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\upsilon_3))} \quad \text{if } y > 1.\end{aligned}$$

Proof Let $f \in W(L^{p_1}, L^{q_1}_{\omega_1})$ and $g \in W(L^{p_2}, L^{q_2}_{\omega_2})$ be given. From Lemma 2.3, we have $D^{p_1}_y f \in W(L^{p_1}, L^{q_1}_{\omega_1})$ and $D^{p_2}_y g \in W(L^{p_2}, L^{q_2}_{\omega_2})$. Also it is known by Theorem 2.5 in [10] that

$$B_{D^q_y m}(f,g)(y) = D^{p_3}_{y^{-1}} B_m\big(D^{p_1}_y f, D^{p_2}_y g\big)(y).$$

If we use this equality, we write

$$\begin{aligned}\big\|B_{D^q_y m}(f,g)\big\|_{W(L^{p_3},L^{q_3}_{\omega_3})} &= \left\| \left\{ \int_{\mathbb{R}^n} \big|D^{p_3}_{y^{-1}} B_m\big(D^{p_1}_y f, D^{p_2}_y g\big)(t)\chi_{Q+x}(t)\big|^{p_3}\,dt \right\}^{\frac{1}{p_3}} \right\|_{q_3,\omega_3} \\ &= \left\| \left\{ \int_{\mathbb{R}^n} \left|y^{\frac{n}{p_3}} B_m\big(D^{p_1}_y f, D^{p_2}_y g\big)\left(\frac{t}{y^{-1}}\right)\chi_{Q+x}(t)\right|^{p_3}\,dt \right\}^{\frac{1}{p_3}} \right\|_{q_3,\omega_3} \\ &= \left\| \left\{ \int_{\mathbb{R}^n} y^n \big|B_m\big(D^{p_1}_y f, D^{p_2}_y g\big)(ty)\chi_{Q+x}(t)\big|^{p_3}\,dt \right\}^{\frac{1}{p_3}} \right\|_{q_3,\omega_3}.\end{aligned}$$

If we say $yt = u$ in the last equality, we have

$$\begin{aligned}\big\|B_{D^q_y m}(f,g)\big\|_{W(L^{p_3},L^{q_3}_{\omega_3})} &= \left\| \left\{ \int_{\mathbb{R}^n} \big|B_m\big(D^{p_1}_y f, D^{p_2}_y g\big)(u)\chi_{yQ+yx}(u)\big|^{p_3}\,dt \right\}^{\frac{1}{p_3}} \right\|_{q_3,\omega_3} \\ &= \big\|F^{yQ}_{B_m(D^{p_1}_y f, D^{p_2}_y g)}(yx)\big\|_{q_3,\omega_3}. \qquad (2.31)\end{aligned}$$

On the other hand, since ω_3 is a slowly increasing weight function, there exist $C_3 > 0$, $N_3 > 0$ such that $\omega_3(x) \le C_3(1+x^2)^{N_3} = \upsilon_3(x)$. If we make the substitution $yx = s$ in equality (2.31), we obtain

$$\begin{aligned}\big\|B_{D^q_y m}(f,g)\big\|_{W(L^{p_3},L^{q_3}_{\omega_3})} &= \big\|F^{yQ}_{B_m(D^{p_1}_y f, D^{p_2}_y g)}(yx)\big\|_{q_3,\omega_3} \\ &= \left\{ \int_{\mathbb{R}^n} \big|F^{yQ}_{B_m(D^{p_1}_y f, D^{p_2}_y g)}(s)\big|^{q_3} \omega_3\big(y^{-1}s\big)^{q_3} y^{-n}\,ds \right\}^{\frac{1}{q_3}} \\ &= y^{-\frac{n}{q_3}} \left\{ \int_{\mathbb{R}^n} \big|F^{yQ}_{B_m(D^{p_1}_y f, D^{p_2}_y g)}(s)\big|^{q_3} \omega_3\big(y^{-1}s\big)^{q_3}\,ds \right\}^{\frac{1}{q_3}} \\ &\le y^{-\frac{n}{q_3}} \left\{ \int_{\mathbb{R}^n} \big|F^{yQ}_{B_m(D^{p_1}_y f, D^{p_2}_y g)}(s)\big|^{q_3} \big(C_3\big(1+y^{-2}s^2\big)^{N_3}\big)^{q_3}\,ds \right\}^{\frac{1}{q_3}}.\end{aligned}$$

We assume that $y \le 1$. Then

$$\begin{aligned}\big\|B_{D^q_y m}(f,g)\big\|_{W(L^{p_3},L^{q_3}_{\omega_3})} &\le y^{-\frac{n}{q_3}} \left\{ \int_{\mathbb{R}^n} \big|F^{yQ}_{B_m(D^{p_1}_y f, D^{p_2}_y g)}(s)\big|^{q_3} \big(C_3\big(y^{-2}+y^{-2}s^2\big)^{N_3}\big)^{q_3}\,ds \right\}^{\frac{1}{q_3}} \\ &= y^{-\frac{n}{q_3}-2N_3} \big\|F^{yQ}_{B_m(D^{p_1}_y f, D^{p_2}_y g)}\big\|_{q_3,\upsilon_3}.\end{aligned}$$

Also, since yQ is a compact set, we have

$$\begin{aligned}\big\|B_{D^q_y m}(f,g)\big\|_{W(L^{p_3},L^{q_3}_{\omega_3})} &\le C y^{-\frac{n}{q_3}-2N_3} \big\|F^{Q}_{B_m(D^{p_1}_y f, D^{p_2}_y g)}\big\|_{q_3,\upsilon_3} \\ &= C y^{-\frac{n}{q_3}-2N_3} \big\|B_m\big(D^{p_1}_y f, D^{p_2}_y g\big)\big\|_{W(L^{p_3},L^{q_3}_{\upsilon_3})}. \qquad (2.32)\end{aligned}$$

Since $m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\upsilon_3)]$, by Lemma 2.3 and inequality (2.32), we obtain

$$\begin{aligned}&\left\|B_{D_y^q m}(f,g)\right\|_{W(L^{p_3},L^{q_3}_{\omega_3})}\\&\quad\le Cy^{-\frac{n}{q_3}-2N_3}\|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\upsilon_3))}\|f\|_{W(L^{p_1},L^{q_1}_{\upsilon_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\upsilon_2})}.\end{aligned} \tag{2.33}$$

Then $D_y^q m \in \mathrm{BM}[W(p_1,q_1,\upsilon_1;p_2,q_2,\upsilon_2;p_3,q_3,\omega_3)]$, and by (2.33) we have

$$\left\|D_y^q m\right\|_{(W(p_1,q_1,\upsilon_1;p_2,q_2,\upsilon_2;p_3,q_3,\omega_3))} \le Cy^{-\frac{n}{q_3}-2N_3}\|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\upsilon_3))}.$$

Now let $y > 1$. Again, since $m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\upsilon_3)]$, by Lemma 2.3 and inequality (2.32), we obtain

$$\begin{aligned}&\left\|B_{D_y^q m}(f,g)\right\|_{W(L^{p_3},L^{q_3}_{\omega_3})}\\&\quad< C\left\|B_m\left(D_y^{p_1}f,D_y^{p_2}g\right)\right\|_{W(L^{p_3},L^{q_3}_{\upsilon_3})}\\&\quad< Cy^{\frac{n}{q_1}+\frac{n}{q_2}+2N_1+2N_2}\|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\upsilon_3))}\|f\|_{W(L^{p_1},L^{q_1}_{\upsilon_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\upsilon_2})}.\end{aligned} \tag{2.34}$$

Thus $D_y^q m \in \mathrm{BM}[W(p_1,q_1,\upsilon_1;p_2,q_2,\upsilon_2;p_3,q_3,\omega_3)]$, and by (2.34) we have

$$\left\|D_y^q m\right\|_{(W(p_1,q_1,\upsilon_1;p_2,q_2,\upsilon_2;p_3,q_3,\omega_3))} < Cy^{\frac{n}{q_1}+\frac{n}{q_2}+2N_1+2N_2}\|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\upsilon_3))}. \qquad \square$$

Theorem 2.6 *Let* $m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$.

(a) *If* $\Phi \in L^1(\mathbb{R}^{2n})$, *then* $\Phi * m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$ *and*

$$\|\Phi * m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))} \le \|\Phi\|_1\|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))}.$$

(b) *If* $\Phi \in L^1_\omega(\mathbb{R}^{2n})$ *such that* $\omega(u,\upsilon) = \omega_1(u)\omega_2(\upsilon)$, *then* $\hat{\Phi}m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$ *and*

$$\|\hat{\Phi}m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))} \le \|\Phi\|_{1,\omega}\|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))}.$$

Proof (a) Let $f,g \in C_c^\infty(\mathbb{R}^n)$ be given. By Proposition 2.5 in [12]

$$B_{\Phi*m}(f,g)(y) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Phi(u,v)B_{T_{(u,v)}m}(f,g)(y)\,du\,dv.$$

If we use Theorem 2.4 and the assumption $m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$, we have

$$\begin{aligned}&\left\|B_{\Phi*m}(f,g)\right\|_{W(L^{p_3},L^{q_3}_{\omega_3})}\\&\quad\le\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left\|\Phi(u,v)B_{T_{(u,v)}m}(f,g)\right\|_{W(L^{p_3},L^{q_3}_{\omega_3})}\,du\,dv\\&\quad\le\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\Phi(u,v)\right|\|T_{(u,v)}m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))}\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}\,du\,dv\\&\quad=\|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))}\|\Phi\|_1\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}<\infty.\end{aligned} \tag{2.35}$$

Hence $\Phi * m \in \mathrm{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$, and by (2.35) we obtain

$$\|\Phi * m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))} \le \|\Phi\|_1 \|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))}.$$

(b) Let $\Phi \in L^1_\omega(\mathbb{R}^{2n})$. Take any $f, g \in C_c^\infty(\mathbb{R}^n)$. It is known by Proposition 2.5 in [12] that

$$B_{\hat{\Phi}m}(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(u,v) B_{M_{(-u,-v)}m}(f,g)(x)\,du\,dv.$$

Since $m \in \mathrm{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$, we have $M_{(-u,-v)}m \in \mathrm{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ and

$$\|M_{(-u,-v)}m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))} \le \omega_1(u)\omega_2(v)\|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))}$$

by Theorem 2.4. Then

$$\begin{aligned}
&\bigl\|B_{\hat{\Phi}m}(f,g)\bigr\|_{W(L^{p_3},L^{q_3}_{\omega_3})} \\
&\quad\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \bigl\|\Phi(u,v)B_{M_{(-u,-v)}m}(f,g)\bigr\|_{W(L^{p_3},L^{q_3}_{\omega_3})}\,du\,dv \\
&\quad\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \bigl|\Phi(u,v)\bigr| \|M_{(-u,-v)}m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))}\|f\|_{p_1,\omega_1}\|g\|_{p_2,\omega_2}\,du\,dv \\
&\quad\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \bigl|\Phi(u,v)\bigr|\omega_1(u)\omega_2(\upsilon)\|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))} \\
&\qquad\times \|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}\,du\,dv \\
&\quad= \|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))}\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}\|\Phi\|_{1,\omega}.
\end{aligned} \tag{2.36}$$

Thus from (2.36) we obtain $\hat{\Phi}m \in \mathrm{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ and

$$\|\hat{\Phi}m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))} \le \|\Phi\|_{1,\omega}\|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))}. \qquad \square$$

Theorem 2.7 *Let $m \in \mathrm{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$. If Q_1, Q_2 are bounded measurable sets in $\mathbb{R}^n$, then*

$$\begin{aligned}
h(\xi,\eta) &= \frac{1}{\mu(Q_1 \times Q_2)} \iint_{Q_1\times Q_2} m(\xi+u,\eta+v)\,du\,dv \\
&\in \mathrm{BM}\bigl[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)\bigr].
\end{aligned}$$

Proof Let $f, g \in C_c^\infty(\mathbb{R}^n)$. We know by Theorem 2.9 in [10] that

$$B_h(f,g)(x) = \frac{1}{\mu(Q_1 \times Q_2)} \iint_{Q_1\times Q_2} B_{T_{(-u,-v)m}}(f,g)(x)\,du\,dv.$$

From Theorem 2.4, we have

$$\begin{aligned}
&\bigl\|B_h(f,g)\bigr\|_{W(L^{p_3},L^{q_3}_{\omega_3})} \\
&\quad\le \frac{1}{\mu(Q_1 \times Q_2)} \iint_{Q_1\times Q_2} \bigl\|B_{T_{(-u,-v)m}}(f,g)\bigr\|_{W(L^{p_3},L^{q_3}_{\omega_3})}\,du\,dv
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\mu(Q_1 \times Q_2)} \iint_{Q_1\times Q_2} \|T_{(-u,-v)m}\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))} \\
&\quad \times \|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}\, du\, dv \\
&= \frac{1}{\mu(Q_1 \times Q_2)} \|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))} \\
&\quad \times \|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})} \mu(Q_1 \times Q_2) \\
&= \|m\|_{(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))} \|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}.
\end{aligned}$$

Hence, we obtain $h(\xi,\eta) \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$. □

Theorem 2.8 *Let ω_3 be a continuous, symmetric, slowly increasing weight function, $\omega(u,v) = \omega_1(u)\omega_2(v)$, $\omega_3 \leq \omega_1$, $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3}$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$, $\frac{1}{p_3} + \frac{1}{p_3'} = 1$, $\frac{1}{q_3} + \frac{1}{q_3'} = 1$ and $q_3' \geq p_3'$. Assume that $\Phi \in L^1_\omega(\mathbb{R}^{2n})$, $\Psi_1 \in L^1_{\omega_1}(\mathbb{R}^n)$ and $\Psi_2 \in L^1_{\omega_2}(\mathbb{R}^n)$. If $m(\xi,\eta) = \hat{\Psi}_1(\xi)\hat{\Phi}(\xi,\eta)\hat{\Psi}_2(\eta)$, then $m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$.*

Proof Take any $f,g,h \in C_c^\infty(\mathbb{R}^n)$. Then, by Theorem 2.10 in [10], we write

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta) m(\xi,\eta)\, d\xi\, d\eta \right| \leq \int_{\mathbb{R}^n} \left| h(y) \tilde{B}_{\hat{\Phi}}(f * \Psi_1, g * \Psi_2)(y) \right| dy.$$

On the other hand, we know that the inequalities

$$\|f * \Psi_1\|_{W(L^{p_1},L^{q_1}_{\omega_1})} \leq C_1 \|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})} \|\Psi_1\|_{1,\omega_1} \tag{2.37}$$

and

$$\|g * \Psi_2\|_{W(L^{p_2},L^{q_2}_{\omega_2})} \leq C_2 \|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})} \|\Psi_2\|_{1,\omega_2} \tag{2.38}$$

hold, where $C_1 > 0$, $C_2 > 0$ by [3]. That means $f * \Psi_1 \in W(L^{p_1}, L^{q_1}_{\omega_1})$ and $g * \Psi_2 \in W(L^{p_2}, L^{q_2}_{\omega_2})$. Also, every constant function is bilinear multiplier of type $(W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3))$ under the given conditions. So, by Theorem 2.6, we have $\hat{\Phi} \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$. Now, if we say that $-y = u$, we have

$$\begin{aligned}
&\left\| \tilde{B}_{\hat{\Phi}}(f * \Psi_1, g * \Psi_2)(y) \right\|_{W(L^{p_3},L^{q_3}_{\omega_3})} \\
&= \left\| \left\{ \int_{\mathbb{R}^n} \left| B_{\hat{\Phi}}(f * \Psi_1, g * \Psi_2)(u) \right|^{p_3} \chi_{-Q-x}(u)\, du \right\}^{\frac{1}{p_3}} \right\|_{q_3,\omega_3} \\
&= \left\| F^{-Q}_{B_{\hat{\Phi}}(f*\Psi_1,g*\Psi_2)}(-x) \right\|_{q_3,\omega_3}.
\end{aligned}$$

In this here, we set $-x = u$ and use ω_3 to be symmetric. Then we have

$$\begin{aligned}
\left\| \tilde{B}_{\hat{\Phi}}(f * \Psi_1, g * \Psi_2)(y) \right\|_{W(L^{p_3},L^{q_3}_{\omega_3})} &\leq C_3 \left\| F^{Q}_{B_{\hat{\Phi}}(f*\Psi_1,g*\Psi_2)} \right\|_{q_3,\omega_3} \\
&= C_3 \left\| B_{\hat{\Phi}}(f * \Psi_1, g * \Psi_2) \right\|_{W(L^{p_3},L^{q_3}_{\omega_3})}
\end{aligned} \tag{2.39}$$

by [5]. Using the Hölder inequality, inequalities (2.37), (2.38), (2.39) and $\hat{\Phi} \in \mathrm{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$, we find

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)\,d\xi\,d\eta\right| \\
&\quad \le \|h\|_{W(L^{p_3'},L^{q_3'}_{\omega_3^{-1}})}\left\|B_{\hat{\Phi}}(f*\Psi_1, g*\Psi_2)\right\|_{W(L^{p_3},L^{q_3}_{\omega_3})} \\
&\quad \le C_1C_2C_3\|h\|_{W(L^{p_3'},L^{q_3'}_{\omega_3^{-1}})}\|B_{\hat{\Phi}}\|\|\Psi_1\|_{1,\omega_1}\|\Psi_2\|_{1,\omega_2}\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}.
\end{aligned}$$

If we say $C = C_1C_2C_3\|B_{\hat{\Phi}}\|\|\Psi_1\|_{1,\omega_1}\|\Psi_2\|_{1,\omega_2}$, then we obtain

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)\,d\xi\,d\eta\right| \le C\|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})}\|h\|_{W(L^{p_3'},L^{q_3'}_{\omega_3^{-1}})}.$$

From Theorem 2.2, we have $m \in \mathrm{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$. □

Theorem 2.9 *Let $1 \le p_1, p_2 \le p \le 2$, $1 \le q_1, q_2 \le r \le 2$, $p_3 \ge p'$, $q_3 \ge r'$ and $q_3' \ge p_3'$ such that $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{2}{p}$ and $\frac{1}{q_3} = \frac{1}{q_1} + \frac{1}{q_2} - \frac{2}{r}$. Assume that ω_3 is a continuous, bounded, symmetric weight function. If $m \in W(L^p(\mathbb{R}^{2n}), L^r(\mathbb{R}^{2n})) \cap L^\infty(\mathbb{R}^{2n})$, then $m \in \mathrm{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$.*

Proof Firstly, we show that $m \in \mathrm{BM}[W(p, r, \omega_1; p, r, \omega_2; \infty, \infty, \omega_3)]$. Take any $f, g, h \in C_c^\infty(\mathbb{R}^n)$. Let $A \times B \subset \mathbb{R}^{2n}$ be a closed and bounded rectangle. Since the definition of $W(L^{p'}(\mathbb{R}^{2n}), L^{r'}(\mathbb{R}^{2n}))$ is independent of the choice of a compact set Q, then, by using Fubini's theorem, we get

$$\begin{aligned}
&\left\|\hat{f}(\xi)\hat{g}(\eta)\right\|_{W(L^{p'}(\mathbb{R}^{2n}),L^{r'}(\mathbb{R}^{2n}))} \\
&\quad = \left\|F^{Q}_{\hat{f}\hat{g}}\right\|_{L^{r'}(\mathbb{R}^{2n})} \le C_1\left\|F^{A\times B}_{\hat{f}\hat{g}}\right\|_{L^{r'}(\mathbb{R}^{2n})} \\
&\quad = C_1\left\|\|\hat{f}\chi_{x+A}\|_{p'}\|\hat{g}\chi_{y+B}\|_{p'}\right\|_{L^{r'}(\mathbb{R}^{2n})} = C_1\left\|F^{A}_{\hat{f}}(x)F^{B}_{\hat{g}}(y)\right\|_{L^{r'}(\mathbb{R}^{2n})} \\
&\quad = C_1\left\|F^{A}_{\hat{f}}\right\|_{r'}\left\|F^{B}_{\hat{g}}\right\|_{r'} = C_1\|\hat{f}\|_{W(L^{p'},L^{r'})}\|\hat{g}\|_{W(L^{p'},L^{r'})}
\end{aligned} \tag{2.40}$$

for some $C_1 > 0$. So, we have $\hat{f}(\xi)\hat{g}(\eta) \in W(L^{p'}(\mathbb{R}^{2n}), L^{r'}(\mathbb{R}^{2n}))$. By using the Hölder inequality, the Hausdorff-Young inequality and equality (2.40), we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)\,d\xi\,d\eta\right| \\
&\quad \le C_1\|h\|_1\left\|\hat{f}(\xi)\hat{g}(\eta)\right\|_{W(L^{p'}(\mathbb{R}^{2n}),L^{r'}(\mathbb{R}^{2n}))}\|m\|_{W(L^p(\mathbb{R}^{2n}),L^r(\mathbb{R}^{2n}))} \\
&\quad \le C_1\|h\|_1\|\hat{f}\|_{W(L^{p'},L^{r'})}\|\hat{g}\|_{W(L^{p'},L^{r'})}\|m\|_{W(L^p(\mathbb{R}^{2n}),L^r(\mathbb{R}^{2n}))} \\
&\quad \le C_1C_2\|h\|_{W(L^1,L^1)}\|f\|_{W(L^p,L^r)}\|g\|_{W(L^p,L^r)}\|m\|_{W(L^p(\mathbb{R}^{2n}),L^r(\mathbb{R}^{2n}))} \\
&\quad \le C_1C_2\|\omega_3\|_\infty\|m\|_{W(L^p(\mathbb{R}^{2n}),L^r(\mathbb{R}^{2n}))}\|f\|_{W(L^p,L^r_{\omega_1})}\|g\|_{W(L^p,L^r_{\omega_2})}\|h\|_{W(L^1,L^1_{\omega_3^{-1}})}
\end{aligned}$$

for some $C_2 > 0$. Therefore $m \in \mathrm{BM}[W(p, r, \omega_1; p, r, \omega_2; \infty, \infty, \omega_3)]$ by Theorem 2.2.

Now, we show that $m \in \mathrm{BM}[W(p,r,\omega_1;1,1,\omega_2;p',r',\omega_3)]$. Again, using Fubini's theorem, we have

$$\begin{aligned}
&\|\hat{f}(-v)\hat{h}(u)\|_{W(L^{p'}(\mathbb{R}^{2n}),L^{r'}(\mathbb{R}^{2n}))} \\
&\quad \le C_3 \big\| \|\hat{f}(-v)\hat{h}(u)\chi_{(x,y)+A\times B}\|_{L^{p'}(\mathbb{R}^{2n})} \big\|_{L^{r'}(\mathbb{R}^{2n})} \\
&\quad = C_3 \big\| \|\hat{f}\chi_{-x-A}\|_{p'} \|\hat{h}\chi_{y+B}\|_{p'} \big\|_{L^{r'}(\mathbb{R}^{2n})} = C_3 \big\| F_{\hat{f}}^{-A}(-x) F_{\hat{h}}^{B}(y) \big\|_{L^{r'}(\mathbb{R}^{2n})} \\
&\quad = C_3 \big\| F_{\hat{f}}^{-A} \big\|_{r'} \big\| F_{\hat{h}}^{B} \big\|_{r'} \le C_3 C_4 \|\hat{f}\|_{W(L^{p'},L^{r'})} \|\hat{h}\|_{W(L^{p'},L^{r'})}
\end{aligned} \tag{2.41}$$

for some $C_3 > 0$, $C_4 > 0$. So, $\hat{f}(-v)\hat{h}(u) \in W(L^{p'}(\mathbb{R}^{2n}), L^{r'}(\mathbb{R}^{2n}))$. Similarly, we have

$$\begin{aligned}
\|m(-v,u+v)\|_{W(L^p(\mathbb{R}^{2n}),L^r(\mathbb{R}^{2n}))} &\le C_5 \big\| F_m^{(-A)\times(A+B)} \big\|_{L^r(\mathbb{R}^{2n})} \\
&\le C_5 C_6 \|m\|_{W(L^p(\mathbb{R}^{2n}),L^r(\mathbb{R}^{2n}))}
\end{aligned} \tag{2.42}$$

for some $C_5 > 0$, $C_6 > 0$. That means $m(-v,u+v) \in W(L^p(\mathbb{R}^{2n}), L^r(\mathbb{R}^{2n}))$. We set $\xi + \eta = u$ and $\xi = -v$. Then, by using the Hölder inequality, the Hausdorff-Young inequality, (2.41) and (2.42), we get

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)\,d\xi\,d\eta \right| \\
&\quad = \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(-v)\hat{g}(u+v)\hat{h}(u)m(-v,u+v)\,du\,dv \right| \\
&\quad \le \|g\|_1 \|\hat{f}(-v)\hat{g}(u)\|_{W(L^{p'}(\mathbb{R}^{2n}),L^{r'}(\mathbb{R}^{2n}))} \|m(-v,u+v)\|_{W(L^p(\mathbb{R}^{2n}),L^r(\mathbb{R}^{2n}))} \\
&\quad \le C_3C_4C_5C_6 \|g\|_1 \|\hat{f}\|_{W(L^{p'},L^{r'})} \|\hat{h}\|_{W(L^{p'},L^{r'})} \|m\|_{W(L^p(\mathbb{R}^{2n}),L^r(\mathbb{R}^{2n}))} \\
&\quad \le C_3C_4C_5C_6C_7 \|\omega_3\|_\infty \|m\|_{W(L^p(\mathbb{R}^{2n}),L^r(\mathbb{R}^{2n}))} \|f\|_{W(L^p,L^r_{\omega_1})} \|g\|_{W(L^1,L^1_{\omega_2})} \|h\|_{W(L^p,L^r_{\omega_3^{-1}})}.
\end{aligned}$$

Thus, by Theorem 2.2, we obtain $m \in \mathrm{BM}[W(p,r,\omega_1;1,1,\omega_2;p',r',\omega_3)]$. Similarly, if we change the variables $\xi + \eta = u$ and $\eta = -v$, then

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)\,d\xi\,d\eta \right| \\
&\quad = \|f\|_1 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\hat{g}(-v)\hat{h}(u)| |m(u+v,-v)|\,du\,dv \\
&\quad \le C \|\omega_3\|_\infty \|m\|_{W(L^p(\mathbb{R}^{2n}),L^r(\mathbb{R}^{2n}))} \|f\|_{W(L^1,L^1_{\omega_1})} \|g\|_{W(L^p,L^r_{\omega_2})} \|h\|_{W(L^p,L^r_{\omega_3^{-1}})}.
\end{aligned}$$

Hence $m \in \mathrm{BM}[W(1,1,\omega_1;p,r,\omega_2;p',r',\omega_3)]$.

We take $\tilde{p}_1$, $\tilde{q}_1$, $\tilde{p}_3$ and $\tilde{q}_3$ such that $1 \le \tilde{p}_1 \le p$, $1 \le \tilde{q}_1 \le r$, $p' \le \tilde{p}_3 \le \infty$ and $r' \le \tilde{q}_3 \le \infty$. Since $m \in \mathrm{BM}[W(p,r,\omega_1;p,r,\omega_2;\infty,\infty,\omega_3)]$ and $m \in \mathrm{BM}[W(1,1,\omega_1;p,r,\omega_2;p',r',\omega_3)]$, we have $m \in \mathrm{BM}[W(\tilde{p}_1,\tilde{q}_1,\omega_1;p,r,\omega_2;\tilde{p}_3,\tilde{q}_3,\omega_3)]$ by the interpolation theorem in [13, 14] such that

$$\frac{1}{\tilde{p}_1} = \frac{1-\theta}{p} + \frac{\theta}{1}, \qquad \frac{1}{\tilde{p}_3} = \frac{1-\theta}{\infty} + \frac{\theta}{p'}, \tag{2.43}$$

$$\frac{1}{\tilde{q}_1} = \frac{1-\theta}{r} + \frac{\theta}{1}, \qquad \frac{1}{\tilde{q}_3} = \frac{1-\theta}{\infty} + \frac{\theta}{r'} \tag{2.44}$$

for all $0 \leq \theta \leq 1$. On the other hand, from equalities (2.43) and (2.44), we obtain the equalities $\frac{1}{\tilde{p}_1} - \frac{1}{\tilde{p}_3} = \frac{1}{p}$ and $\frac{1}{\tilde{q}_1} - \frac{1}{\tilde{q}_3} = \frac{1}{r}$. Similarly, we take $\tilde{p}_2$, $\tilde{q}_2$, $\tilde{r}_3$ and $\tilde{s}_3$ such that $1 \leq \tilde{p}_2 \leq p$, $1 \leq \tilde{q}_2 \leq r$, $p' \leq \tilde{r}_3 \leq \infty$ and $r' \leq \tilde{s}_3 \leq \infty$. Again, if we use $m \in \mathrm{BM}[W(p,r,\omega_1;p,r,\omega_2;\infty,\infty,\omega_3)]$ and $m \in \mathrm{BM}[W(p,r,\omega_1;1,1,\omega_2;p',r',\omega_3)]$, we have $m \in \mathrm{BM}[W(p,r,\omega_1;\tilde{p}_2,\tilde{q}_2,\omega_2;\tilde{r}_3,\tilde{s}_3,\omega_3)]$ by the interpolation theorem in [13, 14] such that

$$\frac{1}{\tilde{p}_2} = \frac{1-\theta}{p} + \frac{\theta}{1}, \qquad \frac{1}{\tilde{r}_3} = \frac{1-\theta}{\infty} + \frac{\theta}{p'}, \tag{2.45}$$

$$\frac{1}{\tilde{q}_2} = \frac{1-\theta}{r} + \frac{\theta}{1}, \qquad \frac{1}{\tilde{s}_3} = \frac{1-\theta}{\infty} + \frac{\theta}{r'} \tag{2.46}$$

for all $0 \leq \theta \leq 1$. So, from equalities (2.45) and (2.46), we have $\frac{1}{\tilde{p}_2} - \frac{1}{\tilde{r}_3} = \frac{1}{p}$ and $\frac{1}{\tilde{q}_2} - \frac{1}{\tilde{s}_3} = \frac{1}{r}$.

Now, we choose $\tilde{p}_1, \tilde{p}_2$ such that $1 \leq \tilde{p}_1 \leq p_1 < p$ and $1 \leq \tilde{p}_2 \leq p_2 < p$. Let these numbers have the following conditions:

$$\frac{1}{p_1} - \frac{1}{p} = (1-\theta)\left(\frac{1}{\tilde{p}_1} - \frac{1}{p}\right), \tag{2.47}$$

$$\frac{1}{p_2} - \frac{1}{p} = (1-\theta)\left(\frac{1}{\tilde{p}_2} - \frac{1}{p}\right) \tag{2.48}$$

for $0 < \theta < 1$. Again, we choose $\tilde{q}_1$, $\tilde{q}_2$ such that $1 \leq \tilde{q}_1 \leq q_1 < r$ and $1 \leq \tilde{q}_2 \leq q_2 < r$. Let these numbers have the following conditions:

$$\frac{1}{q_1} - \frac{1}{r} = (1-\theta)\left(\frac{1}{\tilde{q}_1} - \frac{1}{r}\right), \tag{2.49}$$

$$\frac{1}{q_2} - \frac{1}{r} = (1-\theta)\left(\frac{1}{\tilde{q}_2} - \frac{1}{r}\right) \tag{2.50}$$

for $0 < \theta < 1$. If we use equalities (2.45), (2.46), (2.47), (2.48), (2.49) and (2.50), we write

$$\frac{1}{p_1} = \frac{1-\theta}{\tilde{p}_1} + \frac{\theta}{p}, \qquad \frac{1}{p_2} = \frac{1-\theta}{p} + \frac{\theta}{\tilde{p}_2}, \tag{2.51}$$

$$\frac{1}{q_1} = \frac{1-\theta}{\tilde{q}_1} + \frac{\theta}{r}, \qquad \frac{1}{q_2} = \frac{1-\theta}{r} + \frac{\theta}{\tilde{q}_2}. \tag{2.52}$$

Moreover, using the equalities $\frac{1}{\tilde{p}_1} - \frac{1}{\tilde{p}_3} = \frac{1}{p}$, $\frac{1}{\tilde{p}_2} - \frac{1}{\tilde{r}_3} = \frac{1}{p}$, (2.51) and the assumption $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{2}{p}$, we obtain

$$\frac{1}{p_3} = \frac{1-\theta}{\tilde{p}_3} + \frac{\theta}{\tilde{r}_3}. \tag{2.53}$$

Similarly, using the equalities $\frac{1}{\tilde{q}_1} - \frac{1}{\tilde{q}_3} = \frac{1}{r}$, $\frac{1}{\tilde{q}_2} - \frac{1}{\tilde{s}_3} = \frac{1}{r}$, (2.52) and the assumption $\frac{1}{q_3} = \frac{1}{q_1} + \frac{1}{q_2} - \frac{2}{r}$, we obtain

$$\frac{1}{q_3} = \frac{1-\theta}{\tilde{q}_3} + \frac{\theta}{\tilde{s}_3}. \tag{2.54}$$

Since $m \in \mathrm{BM}[W(\tilde{p}_1,\tilde{q}_1,\omega_1;p,r,\omega_2;\tilde{p}_3,\tilde{q}_3,\omega_3)]$, $m \in \mathrm{BM}[W(p,r,\omega_1;\tilde{p}_2,\tilde{q}_2,\omega_2;\tilde{r}_3,\tilde{s}_3,\omega_3)]$, then the bilinear multipliers $B_m : W(L^{\tilde{p}_1}, L^{\tilde{q}_1}_{\omega_1}) \times W(L^p, L^r_{\omega_2}) \to W(L^{\tilde{p}_3}, L^{\tilde{q}_3}_{\omega_3})$ and B_m :

$W(L^p, L^r_{\omega_1}) \times W(L^{\tilde{p}_2}, L^{\tilde{q}_2}_{\omega_2}) \to W(L^{\tilde{r}_3}, L^{\tilde{s}_3}_{\omega_3})$ are bounded. Also, since $1 \le \tilde{p}_1 \le p_1 < p$, $1 \le \tilde{q}_1 \le q_1 < r$, $1 \le \tilde{q}_2 \le q_2 < r$, $1 \le \tilde{p}_2 \le p_2 < p$, by equalities (2.53) and (2.54) and by the interpolation theorem in [13], $B_m : W(L^{p_1}, L^{q_1}_{\omega_1}) \times W(L^{p_2}, L^{q_2}_{\omega_2}) \to W(L^{p_3}, L^{q_3}_{\omega_3})$ is bounded. That means $m \in \mathrm{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$. This completes the proof. □

3 The bilinear multipliers space BM[$W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)$]

Lemma 3.1 *Let* $1 \le p(x) \le q < \infty$.

(a) *If* ω *is a slowly increasing weight function, then* $C_c^\infty(\mathbb{R}^n)$ *is dense in the weighted variable exponent Wiener amalgam space* $W(L^{p(x)}, L^q_\omega)$.

(b) *If* ω *is a continuous, slowly increasing weight function, then* $C_c^\infty(\mathbb{R}^n)$ *is dense in the weighted variable exponent Wiener amalgam space* $W(L^{p(x)}, L^q_{\omega^{-1}})$.

This lemma can be proved easily by using the proof technique in Lemma 2.1.

Definition 3.1 Let $p_1(x), p_2(x), p_3(x) \in P(\mathbb{R}^n)$, $p_1^* < \infty$, $p_2^* < \infty$, $p_3^* < \infty$, $p_1(x) \le q_1$, $p_2(x) \le q_2$, $1 \le q_3 \le \infty$ and ω_1, ω_2, ω_3 be weight functions on $\mathbb{R}^n$. Assume that ω_1, ω_2 are slowly increasing functions and $m(\xi, \eta)$ is a bounded, measurable function on $\mathbb{R}^n \times \mathbb{R}^n$. Define

$$B_m(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) m(\xi, \eta) e^{2\pi i \langle \xi + \eta, x \rangle} \, d\xi \, d\eta$$

for all $f, g \in C_c^\infty(\mathbb{R}^n)$. m is said to be a bilinear multiplier on $\mathbb{R}^n$ of type $(W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3))$ if there exists $C > 0$ such that

$$\left\| B_m(f, g) \right\|_{W(L^{p_3(x)}, L^{q_3}_{\omega_3})} \le C \|f\|_{W(L^{p_1(x)}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2(x)}, L^{q_2}_{\omega 2})}$$

for all $f, g \in C_c^\infty(\mathbb{R}^n)$. That means B_m extends to a bounded bilinear operator from $W(L^{p_1(x)}, L^{q_1}_{\omega_1}) \times W(L^{p_2(x)}, L^{q_2}_{\omega 2})$ to $W(L^{p_3(x)}, L^{q_3}_{\omega_3})$. We denote by $\mathrm{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]$ the space of all bilinear multipliers of type $(W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3))$ and

$$\|m\|_{(W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3))} = \|B_m\|.$$

The following theorem can be proved easily by using Lemma 3.1 and the technique of proof of Theorem 2.2.

Theorem 3.1 *Let* $\frac{1}{p_3(x)} + \frac{1}{p_3'(x)} = 1$, $\frac{1}{q_3} + \frac{1}{q_3'} = 1$, $q_3' \ge p_3'(x)$, $p_3(-x) = p_3(x)$ *and* ω_3 *be a continuous, symmetric, slowly increasing weight function. Then* $m \in \mathrm{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]$ *if and only if there exists* $C > 0$ *such that*

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\xi + \eta) m(\xi, \eta) \, d\xi \, d\eta \right| \le C \|f\|_{W(L^{p_1(x)}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2(x)}, L^{q_2}_{\omega 2})} \|h\|_{W(L^{p_3'(x)}, L^{q_3'}_{\omega_3^{-1}})}$$

for all $f, g, h \in C_c^\infty(\mathbb{R}^n)$.

Now we will give some properties of the space $\mathrm{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]$. Since some properties of usual Lebesgue spaces are not true in general in the variable exponent Lebesgue spaces, like translation invariance, then also some properties of the spaces $\mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3,q_3,\omega_3)]$ do not hold true in general in the spaces $\mathrm{BM}[W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3)]$.

Theorem 3.2 *If* $m \in \mathrm{BM}[W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3)]$, *then* $T_{(\xi_0,\eta_0)}m \in \mathrm{BM}[W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3)]$ *and*

$$\|T_{(\xi_0,\eta_0)}m\|_{(W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3))} = \|m\|_{(W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3))}$$

for all $(\xi_0,\eta_0) \in \mathbb{R}^{2n}$.

Proof Let $f,g \in C_c^\infty(\mathbb{R}^n)$. Then we have

$$\|M_{-\xi_0}f\|_{W(L^{p_1(x)},L^{q_1}_{\omega_1})} = \left\| \left\| e^{2\pi i\langle -\xi_0,\cdot\rangle} f(\cdot)\chi_{z+Q}(\cdot)\right\|_{p_1(x)}\right\|_{q_1,\omega_1} = \|f\|_{W(L^{p_1(x)},L^{q_1}_{\omega_1})}.$$

Similarly, the equality $\|M_{-\eta_0}g\|_{W(L^{p_2(x)},L^{q_2}_{\omega_2})} = \|g\|_{W(L^{p_2(x)},L^{q_2}_{\omega_2})}$ is written. So, by using these results and the assumption $m \in \mathrm{BM}[W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3)]$, we have

$$\begin{aligned}
&\left\|B_{T_{(\xi_0,\eta_0)}m}(f,g)\right\|_{W(L^{p_3(x)},L^{q_3}_{\omega_3})} \\
&\quad = \left\|B_m(M_{-\xi_0}f,M_{-\eta_0}g)\right\|_{W(L^{p_3(x)},L^{q_3}_{\omega_3})} \\
&\quad \le \|B_m\|\|M_{-\xi_0}f\|_{W(L^{p_1(x)},L^{q_1}_{\omega_1})}\|M_{-\eta_0}g\|_{W(L^{p_2(x)},L^{q_2}_{\omega_2})} \\
&\quad = \|B_m\|\|f\|_{W(L^{p_1(x)},L^{q_1}_{\omega_1})}\|g\|_{W(L^{p_2(x)},L^{q_2}_{\omega_2})}.
\end{aligned}$$

Thus $T_{(\xi_0,\eta_0)}m \in \mathrm{BM}[W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3)]$. Moreover, by using the same technique as in the proof of Theorem 2.4, we obtain

$$\|T_{(\xi_0,\eta_0)}m\|_{(W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3))} = \|m\|_{(W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3))}. \qquad \square$$

Theorem 3.3 *Let* $m \in \mathrm{BM}[W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3)]$. *Then* $\Phi * m \in \mathrm{BM}[W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3)]$, *and there exists* $C>0$ *such that*

$$\|\Phi * m\|_{(W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3))} \le C\|\Phi\|_1\|m\|_{(W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3))}$$

for all $\Phi \in L^1(\mathbb{R}^{2n})$.

Proof Take any $f,g \in C_c^\infty(\mathbb{R}^n)$. By Proposition 2.5 in [12], we know that

$$B_{\Phi*m}(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(u,v)B_{T_{(\xi_u,\eta_v)}m}(f,g)(x)\,du\,dv. \tag{3.1}$$

Since $m \in \mathrm{BM}[W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3)]$, then by Theorem 3.2, $T_{(u,v)}m$ in the space $\mathrm{BM}[W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3)]$ and

$$\|T_{(u,v)}m\|_{(W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3))} = \|m\|_{(W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3))}.$$

Using (3.1) and the Minkowski inequality in [15], we find $C > 0$ such that

$$\begin{aligned}
&\|B_{\Phi*m}(f,g)\|_{W(L^{p_3(x)},L^{q_3}_{\omega_3})} \\
&\quad = \Big\| \Big\| \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(u,v) B_{T_{(u,v)}m}(f,g)\,du\,dv \Big) \chi_{z+Q} \Big\|_{p_3(x)} \Big\|_{q_3,\omega_3} \\
&\quad \le C \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\Phi(u,v)| \big\| \|B_{T_{(u,v)}m}(f,g)\chi_{z+Q}\|_{p_3(x)} \big\|_{q_3,\omega_3}\,du\,dv \\
&\quad = C \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\Phi(u,v)| \|B_{T_{(u,v)}m}(f,g)\|_{W(L^{p_3(x)},L^{q_3}_{\omega_3})}\,du\,dv \\
&\quad \le C \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\Phi(u,v)| \|T_{(u,v)}m\|_{\mathrm{BM}[W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3)]} \\
&\qquad \times \|f\|_{W(L^{p_1(x)},L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2(x)},L^{q_2}_{\omega_2})}\,du\,dv \\
&\quad = C\|m\|_{\mathrm{BM}[W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3)]} \\
&\qquad \times \|f\|_{W(L^{p_1(x)},L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2(x)},L^{q_2}_{\omega_2})} \|\Phi\|_1.
\end{aligned} \tag{3.2}$$

Hence $\Phi * m \in \mathrm{BM}[W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3)]$, and by (3.2) we have

$$\begin{aligned}
&\|\Phi * m\|_{(W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3))} \\
&\quad \le C\|\Phi\|_1 \|m\|_{(W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3))}.
\end{aligned}$$

□

Theorem 3.4 *Let $\frac{1}{p_3(x)} + \frac{1}{p_3'(x)} = 1$, $\frac{1}{q_3} + \frac{1}{q_3'} = 1$, $q_3' \ge p_3'(x)$, $p_3(-x) = p_3(x)$ and ω_3 be a continuous, symmetric, slowly increasing weight function. If $\Psi_1 \in L^1_{\omega_1}(\mathbb{R}^n)$, $\Psi_2 \in L^1_{\omega_2}(\mathbb{R}^n)$ and $m \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3(x),q_3,\omega_3)]$, then $\hat{\Psi}_1(\xi)m(\xi,\eta)\hat{\Psi}_2(\eta) \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3(x),q_3,\omega_3)]$.*

Proof Take any $f,g,h \in C_c^\infty(\mathbb{R}^n)$. Then, by Theorem 2.10 in [10], we write the following:

$$\Big| \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)\,d\xi\,d\eta \Big| \le \int_{\mathbb{R}^n} \big| h(y)\tilde{B}_m(f*\Psi_1, g*\Psi_2)(y) \big|\,dy.$$

So, by Theorem 11.7.1 in [5] and inequalities (2.37), (2.38), there exists $C > 0$ such that

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)\hat{\Psi}_1(\xi)m(\xi,\eta)\hat{\Psi}_2(\eta)\,d\xi\,d\eta \Big| \\
&\quad \le \|h\|_{W(L^{p_3'(x)},L^{q_3'}_{\omega^{-1}})} \|\tilde{B}_m(f*\Psi_1, g*\Psi_2)\|_{W(L^{p_3(x)},L^{q_3}_{\omega_3})} \\
&\quad \le C\|h\|_{W(L^{p_3'(x)},L^{q_3'}_{\omega^{-1}})} \|B_m(f*\Psi_1, g*\Psi_2)\|_{W(L^{p_3(x)},L^{q_3}_{\omega_3})} \\
&\quad \le C\|h\|_{W(L^{p_3'(x)},L^{q_3'}_{\omega^{-1}})} \|B_m\| \|f*\Psi_1\|_{W(L^{p_1},L^{q_1}_{\omega_1})} \|g*\Psi_2\|_{W(L^{p_2},L^{q_2}_{\omega_2})} \\
&\quad \le C\|B_m\| \|\Psi_1\|_{1,\omega_1} \|\Psi_2\|_{1,\omega_2} \|f\|_{W(L^{p_1},L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2},L^{q_2}_{\omega_2})} \|h\|_{W(L^{p_3'(x)},L^{q_3'}_{\omega^{-1}})}.
\end{aligned}$$

Hence, $\hat{\Psi}_1(\xi)m(\xi,\eta)\hat{\Psi}_2(\eta) \in \mathrm{BM}[W(p_1,q_1,\omega_1;p_2,q_2,\omega_2;p_3(x),q_3,\omega_3)]$ by Theorem 3.1. □

Theorem 3.5 *Let* $m \in \mathrm{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]$. *If* $Q_1, Q_2 \subset \mathbb{R}^n$ *are bounded sets, then*

$$\begin{aligned} h(\xi,\eta) &= \frac{1}{\mu(Q_1 \times Q_2)} \iint_{Q_1 \times Q_2} m(\xi + u, \eta + v)\,du\,dv \\ &\in \mathrm{BM}\bigl[W\bigl(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3\bigr)\bigr]. \end{aligned}$$

Proof Let $f, g, h \in C_c^\infty(\mathbb{R}^n)$ be given. The equality

$$B_h(f,g)(x) = \frac{1}{\mu(Q_1 \times Q_2)} \iint_{Q_1 \times Q_2} B_{T_{(-u,-v)}m}(f,g)(x)\,du\,dv$$

is known by Theorem 2.9 in [10]. Using Theorem 3.2, there exists $C > 0$ such that

$$\begin{aligned} &\bigl\|B_h(f,g)\bigr\|_{W(L^{p_3(x)}, L^{q_3}_{\omega_3})} \\ &\quad= \left\| \frac{1}{\mu(Q_1 \times Q_2)} \iint_{Q_1 \times Q_2} B_{T_{(-u,-v)}m}(f,g)\chi_{z+Q}\,du\,dv \right\|_{W(L^{p_3(x)}, L^{q_3}_{\omega_3})} \\ &\quad\le \frac{C}{\mu(Q_1 \times Q_2)} \left\| \iint_{Q_1 \times Q_2} \bigl\|B_{T_{(-u,-v)}m}(f,g)\chi_{z+Q}\bigr\|_{p_3(x)} \right\|_{q_3,\omega_3} du\,dv \\ &\quad\le \frac{C}{\mu(Q_1 \times Q_2)} \iint_{Q_1 \times Q_2} \bigl\| \bigl\|B_{T_{(-u,-v)}m}(f,g)\chi_{z+Q}\bigr\|_{p_3(x)} \bigr\|_{q_3,\omega_3} du\,dv \\ &\quad= \frac{C}{\mu(Q_1 \times Q_2)} \iint_{Q_1 \times Q_2} \bigl\|B_{T_{(-u,-v)}m}(f,g)\bigr\|_{W(L^{p_3(x)}, L^{q_3}_{\omega_3})} du\,dv \\ &\quad\le \frac{1}{\mu(Q_1 \times Q_2)} C \iint_{Q_1 \times Q_2} \|T_{(-u,-v)}m\|_{(W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3))} \\ &\qquad \times \|f\|_{W(L^{p_1(x)}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2(x)}, L^{q_2}_{\omega_2})}\,du\,dv \\ &\quad= C\|m\|_{(W(p_1(x),q_1,\omega_1;p_2(x),q_2,\omega_2;p_3(x),q_3,\omega_3))} \|f\|_{W(L^{p_1(x)}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2(x)}, L^{q_2}_{\omega_2})}. \end{aligned}$$

Hence $h(\xi,\eta) \in m \in \mathrm{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]$. □

Theorem 3.6 *Let* $r(x) \le s(x)$, $m(x) \le q(x)$, $n(x) \le p(x)$, $s \le r$, $q \le m$, $p \le n$, $\upsilon_3 \le \omega_3$, $\omega_2 \le \upsilon_2$, $\omega_1 \le \upsilon_1$. *Then*

$$\begin{aligned} &\mathrm{BM}\bigl[W\bigl(n(x), n, \omega_1; m(x), m, \omega_2; s(x), s, \omega_3\bigr)\bigr] \\ &\quad\subset \mathrm{BM}\bigl[W\bigl(p(x), p, \upsilon_1; q(x), q, \upsilon_2; r(x), r, \upsilon_3\bigr)\bigr]. \end{aligned}$$

Proof Take any $m \in \mathrm{BM}[W(n(x), n, \omega_1; m(x), m, \omega_2; s(x), s, \omega_3)]$. Then there exists $C_1 > 0$ such that

$$\bigl\|B_m(f,g)\bigr\|_{W(L^{s(x)}, L^s_{\omega_3})} \le C_1 \|f\|_{W(L^{n(x)}, L^n_{\omega_1})} \|g\|_{W(L^{m(x)}, L^m_{\omega_2})}. \tag{3.3}$$

On the other hand, by Proposition 2.5 in [7] we have $W(L^{s(x)}, L^s_{\omega_3}) \subset W(L^{r(x)}, L^r_{\upsilon_3})$, $W(L^{p(x)}, L^p_{\upsilon_1}) \subset W(L^{n(x)}, L^n_{\omega_1})$ and $W(L^{q(x)}, L^q_{\upsilon_2}) \subset W(L^{m(x)}, L^m_{\omega_2})$. So, there exist $C_2 > 0$,

$C_3 > 0$ and $C_4 > 0$ such that

$$\left\|B_m(f,g)\right\|_{W(L^{r(x)},L^r_{\upsilon_3})} \leq C_2 \left\|B_m(f,g)\right\|_{W(L^{s(x)},L^s_{\omega_3})}, \tag{3.4}$$

$$\|f\|_{W(L^{n(x)},L^n_{\omega_1})} \leq C_3 \|f\|_{W(L^{p(x)},L^p_{\upsilon_1})} \tag{3.5}$$

and

$$\|g\|_{W(L^{m(x)},L^m_{\omega 2})} \leq C_4 \|g\|_{W(L^{q(x)},L^q_{\upsilon_2})}. \tag{3.6}$$

Combining (3.3), (3.4), (3.5) and (3.6), we get

$$\left\|B_m(f,g)\right\|_{W(L^{r(x)},L^r_{\upsilon_3})} \leq C_1 C_2 C_3 C_4 \|f\|_{W(L^{p(x)},L^p_{\upsilon_1})} \|g\|_{W(L^{q(x)},L^q_{\upsilon_2})}.$$

That means $m \in \mathrm{BM}[W(p(x),p,\upsilon_1;q(x),q,\upsilon_2;r(x),r,\upsilon_3)]$. Hence, we obtain $\mathrm{BM}[W(n(x),n,\omega_1;m(x),m,\omega_2;s(x),s,\omega_3)] \subset \mathrm{BM}[W(p(x),p,\upsilon_1;q(x),q,\upsilon_2;r(x),r,\upsilon_3)]$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Faculty of Arts and Sciences, Ondokuz Mayıs University, Kurupelit, Samsun, 55139, Turkey.
[2]Department of Mathematics and Computer Sciences, Faculty of Science and Letters, İstanbul Arel University, Tepekent, Büyükçekmece, İstanbul, Turkey.

Acknowledgements
This work was supported by the Ondokuz Mayıs University, Project number PYO.FEN.1904.13.002.

Received: 9 September 2014 Accepted: 12 November 2014 Published: 27 Nov 2014

References
1. Gasquet, C, Witomski, P: Fourier Analysis and Applications. Springer, New York (1999)
2. Rudin, W: Fourier Analysis on Groups. Interscience, New York (1962)
3. Feichtinger, HG: Banach convolution algebras of Wiener type. In: Proc. Conf. Functions, Series, Operators, Budapest. Colloq. Math. Soc. Janos Bolyai, vol. 35, pp. 509-524 (1980)
4. Fischer, RH, Gürkanlı, AT, Liu, TS: On a family of Wiener type spaces. Int. J. Math. Sci. **19**(1), 57-66 (1996)
5. Heil, C: An introduction to weighted Wiener amalgams. In: Wavelets and Their Applications, pp. 183-216. Allied Publishers, New Delhi (2003)
6. Kovacik, O, Rakosnik, J: On spaces $L^{p(x)}$ and $W^{k,p(x)}$. Czechoslov. Math. J. **41(116)**(4), 592-618 (1991)
7. Aydın, İ, Gürkanlı, AT: Weighted variable exponent amalgam spaces. Glas. Mat. **47**(67), 165-174 (2012)
8. Murthy, GNK, Unni, KR: Multipliers on weighted spaces. In: Functional Analysis and Its Applications (International Conference). Lecture Notes in Mathematics, vol. 399, pp. 272-291. Springer, Berlin (1973)
9. Dobler, T: Wiener Amalgam Spaces on Locally Compact Groups. Master's thesis, University of Vienna (1989)
10. Kulak, Ö, Gürkanlı, AT: Bilinear multipliers of weighted Lebesgue spaces and variable exponent Lebesgue spaces. J. Inequal. Appl. **2013**, 259 (2013)
11. Feichtinger, HG: A characterization of Wiener's algebra on locally compact groups. Arch. Math. (Basel) **29**, 136-140 (1977)
12. Blasco, O: Notes on the spaces of bilinear multipliers. Rev. Unión Mat. Argent. **50**(2), 23-37 (2009)
13. Bennett, C, Sharpley, R: Interpolation of Operators. Academic Press, San Diego (1988)
14. Feichtinger, HG: Banach spaces of distributions of Wiener's type and interpolation. In: Proc. Conf. Funct. Anal. Approx., Oberwolfach, vol. 60, pp. 153-165. Birkhäuser, Basel (1981)
15. Samko, SG: Convolution type operators in $L^{p(x)}$. Integral Transforms Spec. Funct. **7**(1-2), 123-144 (1998)

10.1186/1029-242X-2014-476
Cite this article as: Kulak and Gürkanlı: **Bilinear multipliers of weighted Wiener amalgam spaces and variable exponent Wiener amalgam spaces.** *Journal of Inequalities and Applications* 2014, 2014:476