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ON THE WEIGHTED VARIABLE EXPONENT AMALGAM SPACE $W(L^{p(x)}, L^q_m)$*

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Abstract In [4], a new family $W(L^{p(x)}, L^q_m)$ of Wiener amalgam spaces was defined and investigated some properties of these spaces, where local component is a variable exponent Lebesgue space $L^{p(x)}(\mathbb{R})$ and the global component is a weighted Lebesgue space $L^q_m(\mathbb{R})$. This present paper is a sequel to our work [4]. In Section 2, we discuss necessary and sufficient conditions for the equality $W\left(L^{p(x)}, L^q_m\right) = L^q(\mathbb{R})$. Later we give some characterization of Wiener amalgam space $W\left(L^{p(x)}, L^q_m\right)$. In Section 3 we define the Wiener amalgam space $W\left(\mathcal{F}L^{p(x)}, L^q_m\right)$ and investigate some properties of this space, where $\mathcal{F}L^{p(x)}$ is the image of $L^{p(x)}$ under the Fourier transform. In Section 4, we discuss boundedness of the Hardy-Littlewood maximal operator between some Wiener amalgam spaces.

Key words weighted Lebesgue space; variable exponent Lebesgue

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1 Introduction

Function spaces with variable exponents have been intensively studied in recent years by a significant number of authors. The variable exponent Lebesgue spaces (or generalized Lebesgue spaces) $L^{p(x)}$ appeared in literature for the first time already in a 1931 article by Orlicz [28], but the modern development started with the paper [27] of Kovacik and Rakosnik in 1991. A survey of the history of the field with a bibliography of more than a hundred titles published up to 2004 can be found in [10]; further surveys are due to Samko [33] and Kokilashvili [26]. The boundedness of the maximal operator was an open problem in $L^{p(x)}$ for a long time. It was first proved by L. Diening over bounded domains in 2004 [9]. After this paper, many interesting and important papers appeared in non-weighted and weighted variable exponent

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spaces. The area which is now called variable exponent analysis, last decade became a rather branched field with many interesting results obtained in Harmonic Analysis, Approximation Theory, Operator Theory (maximal, singular operators and potential type operators), Pseudo-Differential Operators.

The interest on Lebesgue spaces with variable exponent comes not only from their own theoretical curiosity but also from their importance in some applications, the motivation to study such function spaces comes from applications to fluid dynamics ([1], [31]), image processing [6], PDE and the calculus of variation ([2], [13]).

2 Notations

Throughout this paper we denote by $C_c(\mathbb{R}), C_0^{\infty}(\mathbb{R}), C_0(\mathbb{R})$ and $S(\mathbb{R})$ the space of complex-valued continuous function on $\mathbb{R}$ with compact support, the space of complex-valued continuous functions on $\mathbb{R}$ which are continuously differentiable many times and have compact support, the space of complex-valued continuous function on $\mathbb{R}$ vanish at infinity and the Schwartz space on $\mathbb{R}$ respectively. We also denote by $S'(\mathbb{R})$ its topological dual. For any measurable function $f:\mathbb{R}\to\mathbb{C}$, the translation and modulation operators T_x and M_w are given by $T_xf(t)=f(t-x)$ and $M_wf(t)=\mathrm{e}^{\mathrm{i}wt}f(t)$. A weight function m on $\mathbb{R}$ is a non-negative, continuous and locally integrable function. m is called submultiplicative if $m(x+y)\le m(x)m(y)$ for all $x,y\in\mathbb{R}$. Let v be a submultiplicative function on $\mathbb{R}$. A weight function m on $\mathbb{R}$ is $v-$moderate if $m(x+y)\le Cv(x)m(y)$ for all $x,y\in\mathbb{R}$. Let m_1 and m_2 be two weights. We say that $m_2\preceq m_1$ if there exists $C>0$ such that $m_2(x)\le Cm_1(x)$ for all $x\in\mathbb{R}$. Two weight functions m_1 and m_2 are called equivalent and we write $m_1\approx m_2$, if $m_2\preceq m_1$ and $m_1\preceq m_2$, [18], [30]. It is easy to see that equivalent weights (also moderate functions) define the same weighted spaces, given by

$$L_m^q(\mathbb{R})=\{f: fm\in L^q\}$$

with the natural norm

$$\|f\|_{L_m^q}=\|fm\|_{L^q}=\left\{\int_{\mathbb{R}}|f(x)m(x)|^q\,\mathrm{d}x\right\}^{\frac{1}{q}},\quad 1\le q<\infty$$

or

$$\|f\|_{L_m^\infty}=\|fm\|_{L^\infty}=\operatorname*{ess\,sup}_{x\in\mathbb{R}}|f(x)|\,m(x),\quad q=\infty.$$

It is known that $\left(L_m^q(\mathbb{R}),\|.\|_{L_m^q}\right)$ is a Banach space and the dual of the space $L_m^q(\mathbb{R})$ is the space $L_{m^{-1}}^s(\mathbb{R})$, where $1\le q<\infty$, $\frac{1}{q}+\frac{1}{s}=1$. It is well known that if m is submultiplicative and $m(x)\ge 1$ for all $x\in\mathbb{R}^n$, then $\left(L_m^1(\mathbb{R}),\|.\|_{L_m^1}\right)$ is a Banach algebra with respect to convolution. It is called Beurling algebra [17], [19], [30] and [34].

Let $f\in L^1(\mathbb{R})$. The Fourier transform $\widehat{f}$ (or $\mathcal{F}f$) of f is given by

$$\widehat{f}(t)=\int_{\mathbb{R}}f(x)\,\mathrm{e}^{-2\pi\mathrm{i}xy}\mathrm{d}x$$

for $x,t\in\mathbb{R}$.

Let $p:\mathbb{R}\to[1,\infty)$ be a measurable function (called the variable exponent on $\mathbb{R}$). We put

$$p_*=\operatorname*{ess\,inf}_{x\in\mathbb{R}}p(x),\qquad p^*=\operatorname*{ess\,sup}_{x\in\mathbb{R}}p(x).$$

The variable exponent Lebesgue spaces (or generalized Lebesgue space) $L^{p(x)}(\mathbb{R})$ is defined to be the space of all measurable functions (equivalent classes) f on $\mathbb{R}$ such that

$$\varrho_p(f)=\int_{\mathbb{R}}|\lambda f(x)|^{p(x)}\,\mathrm{d}x<\infty$$

for some $\lambda=\lambda(f)>0$. The function $\varrho_p(f)$ called modular of the space $L^{p(x)}(\mathbb{R})$. Then

$$\|f\|_{L^{p(x)}}=\inf\left\{\lambda>0:\varrho_p(\frac{f}{\lambda})\leq 1\right\}, \tag{2.1}$$

is a norm (Luxemburg norm) on $L^{p(x)}(\mathbb{R})$. This makes $L^{p(x)}(\mathbb{R})$ a Banach space. If $p(x)=p$ is a constant function, then the norm $\|.\|_{L^{p(x)}}$ coincides with the usual Lebesgue norm $\|.\|_{L^p}$ (see [11], [12], [21], [22], [27]). It is also known that if $p^*<\infty$ then $L^{p(x)}(\mathbb{R})$ is solid space, that is, if any measurable function g, for which there exists $f\in L^{p(x)}(\mathbb{R})$ such that $|g(x)|\leq|f(x)|$ locally almost everywhere, belongs to $L^{p(x)}(\mathbb{R})$, with $\|g\|_{L^{p(x)}}\leq\|f\|_{L^{p(x)}}$, see [3], [5].

A Banach function space (shortly BF-space) on $\mathbb{R}$ is a Banach space $(B,\|.\|_B)$ of measurable functions which is continously embedded into $L^1_{\mathrm{loc}}(\mathbb{R})$, that is for every compact subset $K\subset\mathbb{R}$ there exist some constant $C_K>0$ such that $\|f\chi_K\|_{L^1}\leq C_K\|f\|_B$ for all $f\in B$.

We say that a Banach space X is continuously embedded into a Banach space Y, $X\hookrightarrow Y$, if $X\subset Y$ and there exists a constant $C>0$ such that $\|x\|_X\leq\|x\|_Y$ for all $x\in X$

Let B_1,B_2,B_3 be Banach spaces of functions defined on $\mathbb{R}$. The triple (B_1,B_2,B_3) will be called a Banach convolution triple (BCT), if convolution, given by

$$f_1*f_2(x)=\int_{\mathbb{R}}f_1(x-t)\,f_2(t)\,\mathrm{d}t \tag{2.2}$$

for $f_i\in C_c(\mathbb{R})\cap B_i$ $(i=1,2)$ extends to a bounded, bilinear map from $B_1\times B_2$ into B_3. It is clear that (A,A,A) is (BCT) for some $A\subset L^1(\mathbb{R})$ if and only if A is a Banach convolution algebra.

Research on Wiener amalgam space was initiated by Wiener in [35]. A number of authors worked on amalgam spaces or some special cases of these spaces. But the first systematic study of these spaces was undertaken by Holland [24], [25] and by Fournier, Stewart [20]. The amalgam of L^p and l^q on the real line is the space $W(L^p,l^q)(\mathbb{R})$ consisting of functions f which are locally in L^p and have l^q behavior at infinity in the sense that the norms over $[n,n+1]$ form an l^q-sequence. For $1\leq p,q\leq\infty$ the norm

$$\|f\|_{p,q}=\left[\sum_{n=-\infty}^{\infty}\left[\int_n^{n+1}|f(x)|^p\,\mathrm{d}x\right]^{\frac{q}{p}}\right]^{\frac{1}{q}}<\infty \tag{2.3}$$

makes $W(L^p,l^q)$ into a Banach space. If $p=q$ then $W(L^p,l^q)$ reduces to L^p. A comprehensive general theory of amalgam space $W(B,C)$ on a locally compact group was introduced and studied by Feichtinger in [14–16]. Here B and C are Banach spaces satisfying certain conditions. Fournier and Stewart gave a good historical background of amalgams, see [20].

Throughout this work we will assume that $p^*<\infty$.

3 Some Properties of the Space $W\left(L^{p(x)},L^q_m\right)$

Let $p:\mathbb{R}\to[1,\infty)$ be a measurable function, $1\leq q\leq\infty$ and let m be a weight function $\mathbb{R}$. The variable exponent Wiener amalgam space $W\left(L^{p(x)},L^q_m\right)$ firstly defined and investigated some properties in [4]. This section of this work is a sequel to the paper [4]. In this section we will discuss some more properties of $W\left(L^{p(x)}, L^q_m\right)$.

Definition 3.1 The space $L^{p(x)}_{\text{loc}}(\mathbb{R})$ consists of all (classes of) measurable functions f on $\mathbb{R}$ such that $f\chi_K \in L^{p(x)}(\mathbb{R})$ for every compact subset $K \subset \mathbb{R}$, where χ_K is the characteristic function of K. Let fix an open set $Q \subset \mathbb{R}$ with compact closure. The Wiener amalgam space $W\left(L^{p(x)}, L^q_m\right)$ consists of all elements $f \in L^{p(x)}_{\text{loc}}(\mathbb{R})$ such that $F_f(z) = \|f\chi_{z+Q}\|_{L^{p(x)}}$ belongs to $L^q_m(\mathbb{R})$; the norm of $W\left(L^{p(x)}, L^q_m\right)$ is

$$\|f\|_{W\left(L^{p(x)},L^q_m\right)} = \|F_f\|_{L^q_m}.$$

It is known that the definition of $W\left(L^{p(x)}, L^q_m\right)$ is independent of choice of Q, i.e., different choices of Q define the same space with equivalent norms. Also it is a Banach space with this norm (see Theorem 2.1 in [4]).

Proposition 3.2 Let $p(x)$ and $r(x)$ be variable exponents on $\mathbb{R}$. Then $W\left(L^{p(x)}, L^q_m\right) \subseteq W\left(L^{r(x)}, L^q_m\right)$ if and only if $r(x) \le p(x)$, $x \in \mathbb{R}$.

Proof It is known by Proposition 2.5 in [4] that if $r(x) \le p(x)$ then $W\left(L^{p(x)}, L^q_m\right) \subseteq W\left(L^{r(x)}, L^q_m\right)$. Conversely assume that $W\left(L^{p(x)}, L^q_m\right) \subseteq W\left(L^{r(x)}, L^q_m\right)$. If $r(x) \nleq p(x)$ then $r(x) > p(x)$ or $r(x) \ngtr p(x)$, $r(x) \nless p(x)$. If $r(x) > p(x)$ then by Proposition 2.5 in [4] we write $W\left(L^{r(x)}, L^q_m\right) \subseteq W\left(L^{p(x)}, L^q_m\right)$, a contradiction. If $r(x) \ngtr p(x)$, $r(x) \nless p(x)$, then again by Proposition 2.5 in [4] we have $W\left(L^{p(x)}, L^q_m\right) \nsubseteq W\left(L^{r(x)}, L^q_m\right)$, a contradiction. This completes the proof.

The proof of the following corollary is easy by Proposition 3.2. □

Corollary 3.3 Let $p(x)$ and $r(x)$ be variable exponents on $\mathbb{R}$. Then

$$W\left(L^{p(x)}, L^q_m\right) = W\left(L^{r(x)}, L^q_m\right)$$

if and only if $r(x) = p(x)$.

Proposition 3.4 Let $p(x)$ be variable exponents on $\mathbb{R}$ and $1 \le q \le \infty$. Then $W(L^{p(x)}, L^q_m) = L^q(\mathbb{R})$ if and only if $p(x) = q$ and $m \approx C$, where C is a constant.

Proof Assume that $W\left(L^{p(x)}, L^q_m\right) = L^q(\mathbb{R})$. It is easy to see that $L^q_m(\mathbb{R}) = L^q(\mathbb{R})$ if and only if $m \approx C$, where C is a constant. It is also known by Proposition 11.5.2 in [23] that $W(L^q, L^q_m) = L^q_m(\mathbb{R})$. Hence $W(L^q, L^q_m) = L^q(\mathbb{R})$ if and only $m \approx C$. Finally by assumption and Corollary 1,

$$W\left(L^{p(x)}, L^q_m\right) = L^q(\mathbb{R}) = W(L^q, L^q).$$

if and only if $p(x) = q$ and $m \approx C$. □

Definition 3.5 A family of functions $\{\psi_i\}_{i\in I}$ on $\mathbb{R}$ is called a bounded uniform partition of unity, or BUPU, if

a) $\sum_{i\in I} \psi_i \equiv 1$,

b) $\sup \|\psi_i\|_{L^\infty} < \infty$,

c) there exists a compact set $U \subset \mathbb{R}$ with nonemty interior and points $y_i \in \mathbb{R}$ such that $\operatorname{supp}\psi_i \subset U + y_i$ for all i and

d) for each compact subset $K \subset \mathbb{R}$,

$$\sup_{x\in\mathbb{R}} \#\{i \in I : x \in K + y_i\} = \sup_{i\in I} \#\{j \in I : K + y_i \cap K + y_j\} < \infty.$$

The proof of the following theorem is same as the proof of Theorem 11.6.2 in [23].

Theorem 3.6 Let m be a moderate weight. If $\{\psi_i\}_{i\in I}$ is a BUPU and V is a compact set containing U, then

$$\|f\|_{W\left(L^{p(x)},L^q_m\right)} \approx \left\|\sum_{i\in I}\|f\psi_i\|_{L^{p(x)}}\,\chi_{V+y_i}\right\|_{L^q_m}.$$

Proposition 3.7 Let m be a moderate weight. Set $U=[0,1)$. Then

$$\begin{aligned}\|f\|_{W\left(L^{p(x)},L^q_m\right)} &\approx \left(\sum_{n\in\mathbb{Z}}\|f\chi_{U+n}(.)\|^q_{L^{p(.)}}\,m(z_n)^q\right)^{\frac{1}{q}}\\ &=\left\|\{\|f\chi_{U+n}(.)\|_{L^{p(.)}}\}_{n\in\mathbb{Z}}\right\|_{l^q_\omega}=\left\|\left\{\left\|f\chi_{[n,n+1)}(.)\right\|_{L^{p(.)}}\right\}_{n\in\mathbb{Z}}\right\|_{l^q_\omega},\end{aligned}$$

where ω is the weight function on the indexed set $\mathbb{Z}$ defined by $\omega(n)=m(z_n)$, $z_n\in V+n$.

Proof Since $\{U+n\}_{n\in\mathbb{Z}}$ is a partition of $\mathbb{R}$, it is easy to show that $\{\chi_{U+n}\}_{n\in\mathbb{Z}}$ is a BUPU. If we set $V=[0,1]$ then by Theorem 3.6 we have

$$\begin{aligned}\|f\|_{W\left(L^{p(x)},L^q_m\right)} &= \left\|\sum_{n\in\mathbb{Z}}\|f\chi_{U+n}(.)\|_{L^{p(.)}}\,\chi_{V+n}(x)\right\|_{L^q_m}\\ &=\left(\int_{\mathbb{R}}\left|\sum_{n\in\mathbb{Z}}\|f\chi_{U+n}(.)\|^q_{L^{p(.)}}\,\chi_{V+n}(x)\right|^q m(x)^q\,\mathrm{d}x\right)^{\frac{1}{q}}\\ &=\left(\sum_{n\in\mathbb{Z}}\|f\chi_{U+n}(.)\|^q_{L^{p(.)}}\int_{\mathbb{R}}\chi_{V+n}(x)\,m(x)^q\,\mathrm{d}x\right)^{\frac{1}{q}}\\ &=\left(\sum_{n\in\mathbb{Z}}\|f\chi_{U+n}(.)\|^q_{L^{p(.)}}\int_{V+n}m(x)^q\,\mathrm{d}x\right)^{\frac{1}{q}}.\end{aligned}$$

Since $m(x)^q$ is moderate , by Proposition 11.2.4 in [23] the values

$$\int_{V+n}m(x)^q\,\mathrm{d}x$$

are uniformly equivalent to the values of m^q at any point $z_n\in V+n$. That means there exist constants $C_1(V)>0$ and $C_2(V)>0$ such that

$$C_1(V)\,m^q(z_n)\le\int_{V+n}m(x)^q\,\mathrm{d}x\le C_2(V)\,m^q(z_n)$$

for any $z_n\in V+n$. Thus

$$\begin{aligned}\|f\|_{W\left(L^{p(x)},L^q_m\right)} &\approx \left(\sum_{n\in\mathbb{Z}}\|f\chi_{U+n}\|^q_{L^{p(x)}}\,m(z_n)^q\right)^{\frac{1}{q}}=\left\|\{\|f\chi_{U+n}\|_{L^{p(x)}}\}_{n\in\mathbb{Z}}\right\|_{l^q_\omega}\\ &=\left\|\left\{\left\|f\chi_{[n,n+1)}\right\|_{L^{p(x)}}\right\}_{n\in\mathbb{Z}}\right\|_{L^q_\omega},\end{aligned}$$

where ω is the weight function on the indexed set $\mathbb{Z}$ defined by $\omega(n)=m(z_n)$. □

4 The Space $W\left(\mathcal{F}L^{p(x)}, L^q_m\right)$

Let $\mathcal{F}L^{p(x)}(\mathbb{R})$ be the image of $L^{p(x)}(\mathbb{R})$ under the Fourier transform $\mathcal{F}$. In this section firstly we will investigate some properties of $\mathcal{F}L^{p(x)}(\mathbb{R})$. Later by using this space we will define and discuss some properties of the Wiener amalgam space $W\left(\mathcal{F}L^{p(x)}, L^q_m\right)$.

Proposition 4.1 The space $L^{p(x)}(\mathbb{R})$ is continuously embedded into the space $S'(\mathbb{R})$ of tempered distributions.

Proof Assume that $\frac{1}{p(x)}+\frac{1}{q(x)}=1$. It is known by Theorem 2.11 in [27] that $C_0^\infty(\mathbb{R})$ is dense in $L^{q(x)}(\mathbb{R})$. Now let $f\in S(\mathbb{R})$ be given. There exists $M>1$ such that

$$\varrho_q(f)=\int_{\mathbb{R}}|f(x)|^{q(x)}\,\mathrm{d}x\le M\,\|f\|_{L^1}. \tag{4.1}$$

Since $p^*<\infty$, by the inequality (4) the identity mapping $I: S(\mathbb{R})\to L^{q(x)}(\mathbb{R})$ is continuous. Hence $S(\mathbb{R})$ is continuously embedded into $L^{q(x)}$. Also since $C_0^\infty(\mathbb{R})\subset S(\mathbb{R})$ and $C_0^\infty(\mathbb{R})$ is dense in $L^{q(x)}(\mathbb{R})$, then the Schwartz space $S(\mathbb{R})$ is dense in $L^{q(x)}(\mathbb{R})$. This implies $L^{p(x)}=\left(L^{q(x)}\right)'\hookrightarrow S'(\mathbb{R})$. □

Definition 4.2 The space $\mathcal{F}L^{p(x)}(\mathbb{R})$ is defined by

$$\mathcal{F}L^{p(x)}(\mathbb{R})=\left\{\widehat{f}: f\in L^{p(x)}(\mathbb{R})\right\},$$

where $\mathcal{F}f$ is the Fourier transform of $f\in L^{p(x)}(\mathbb{R})$ in the sense of tempered distribution. We endow this space with the norm

$$\left\|\widehat{f}\right\|_{\mathcal{F}L^{p(x)}}=\|f\|_{L^{p(x)}}. \tag{4.2}$$

Proposition 4.3 (i) The normed space $\left(\mathcal{F}L^{p(x)}(\mathbb{R}),\|.\|_{\mathcal{F}L^{p(x)}}\right)$ is a strongly translation invariant Banach space.

(ii) The translation operator $t\to T_t\widehat{f}$ is continuous from $\mathbb{R}$ into $\mathcal{F}L^{p(x)}(\mathbb{R})$ for all $\widehat{f}\in\mathcal{F}L^{p(x)}(\mathbb{R})$.

Proof (i) Let $\left(\widehat{f_n}\right)$ be a Cauchy sequence in $\mathcal{F}L^{p(x)}(\mathbb{R})$. Then (f_n) is a Cauchy sequence in $L^{p(x)}(\mathbb{R})$. Since $L^{p(x)}(\mathbb{R})$ is a Banach space, (f_n) converges to a function $f\in L^{p(x)}(\mathbb{R})$. Hence given any $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that

$$\left\|\widehat{f}_n-\widehat{f}\right\|_{\mathcal{F}L^{p(x)}}=\|f_n-f\|_{L^{p(x)}}<\varepsilon.$$

for all $n\ge n_0$. Also $\widehat{f}\in\mathcal{F}L^{p(x)}(\mathbb{R})$. Hence $\mathcal{F}L^{p(x)}(\mathbb{R})$ is a Banach space.

Now let $\widehat{f}\in\mathcal{F}L^{p(x)}(\mathbb{R})$ and $t\in\mathbb{R}$ be given. It is known by Lemma 5 in [3] that $L^{p(x)}(\mathbb{R})$ is strongly character invariant. Hence

$$\|M_tf\|_{L^{p(x)}}=\|f\|_{L^{p(x)}}. \tag{4.3}$$

Thus by (4.3)

$$\begin{aligned}\left\|T_t\widehat{f}\right\|_{\mathcal{F}L^{p(x)}}&=\left\|(M_tf)^\wedge\right\|_{\mathcal{F}L^{p(x)}}=\|M_tf\|_{L^{p(x)}}\\&=\|f\|_{L^{p(x)}}=\left\|\widehat{f}\right\|_{\mathcal{F}L^{p(x)}}.\end{aligned}$$

Then $\mathcal{F}L^{p(x)}(\mathbb{R})$ is strongly translation invariant.

(ii) Let any $\widehat{f} \in \mathcal{F}L^{p(x)}(\mathbb{R})$ be given. Since the space $\mathcal{F}L^{p(x)}(\mathbb{R})$ is translation invariant, for the proof of (ii) it is enough to show that the operator $t \to T_t\widehat{f}$ is continuous at $t = 0$. We know by Lemma 5 in [3] that the function $t \to M_t g$ is continuous from $\mathbb{R}$ into $L^{p(x)}(\mathbb{R})$. Then the right side of the equality

$$\left\|T_t\widehat{f} - \widehat{f}\right\|_{\mathcal{F}L^{p(x)}} = \|M_t f - f\|_{L^{p(x)}}$$

tends to zero as $t \to 0$. This completes the proof. □

Corollary 4.4 By Proposition 4.3, $\mathcal{F}L^{p(x)}(\mathbb{R})$ is a Banach space of homogeneous tempered distributions.

Proposition 4.5 $\mathcal{F}L^{p(x)}(\mathbb{R})$ is a Banach convolution module over $\mathcal{F}L^{\infty}(\mathbb{R})$, where $\mathcal{F}L^{\infty}(\mathbb{R})$ is the image of $L^{\infty}(\mathbb{R})$ under the Fourier transform $\mathcal{F}$.

Proof It is known by Lemma 1 in [3] that $L^{p(x)}(\mathbb{R})$ is a Banach module over $L^{\infty}(\mathbb{R})$ with respect to pointwise multiplication.

Now let $(F, G) \in \mathcal{F}L^{\infty}(\mathbb{R}) \times \mathcal{F}L^{p(x)}(\mathbb{R})$. Then there exists $f \in L^{\infty}(\mathbb{R})$ and $g \in L^{p(x)}(\mathbb{R})$ such that $\widehat{f} = F$ and $\hat{g} = G$. Since $L^{p(x)}(\mathbb{R})$ is solid, by Lemma 1 in [3] we obtain

$$\begin{aligned}\|F * G\|_{\mathcal{F}L^{p(x)}} &= \|fg\|_{L^{p(x)}} \le \|f\|_{L^{\infty}} \|g\|_{L^{p(x)}} \\ &= \|F\|_{\mathcal{F}L^{\infty}} \|G\|_{\mathcal{F}L^{p(x)}} < \infty.\end{aligned}$$

Let m be a submultiplicative weight function on $\mathbb{R}$. Since $W\left(C_0, \ell^1_m\right)$ is a Banach convolution algebra, then the image $A = \mathcal{F}\left(W\left(C_0, \ell^1_m\right)\right)$ of the space $W\left(C_0, \ell^1_m\right)$ under Fourier transform is a Banach algebra with respect to pointwise multiplication [14]. □

Proposition 4.6 Let m be a submultiplicative weight function on $\mathbb{R}$. Then $A = \mathcal{F}W(C_0, \ell^1)$ has a bounded uniform partition of unity (BUPU).

Proof Take the characteristic function $\chi_{[-\frac{1}{2},\frac{1}{2}]}$ and a function $g \in S(\mathbb{R}), g(t) \ge 0$ with compact support such that $\int_{-\infty}^{+\infty} g(t)\,\mathrm{d}t = 1$. It is easy to show that $g * 1 = 1$. Define $\varphi_n(t) = \chi_{n+[-\frac{1}{2},\frac{1}{2}]}(t) = T_n\chi_{[-\frac{1}{2},\frac{1}{2}]}(t)$, $n \in \mathbb{Z}$. Then $\sum\limits_{n\in\mathbb{Z}} \varphi_n(t) = 1$. Since $g \in S(\mathbb{R})$, then $g * \varphi_n \in S(\mathbb{R})$ for all $n \in \mathbb{Z}$. If we set $\psi_n = g * \varphi_n$, the family $\{\psi_n\}_{n\in\mathbb{Z}}$ is a Bounded uniform partition of unity (BUPU) on $\mathbb{R}$. Indeed

$$\begin{aligned}1 = (g * 1)(x) &= \left(g * \sum_{n\in\mathbb{Z}} \varphi_n\right)(x) = \left(g * \sum_{n\in\mathbb{Z}} T_n\chi_{[-\frac{1}{2},\frac{1}{2}]}\right)(x) \\ &= \sum_{n\in\mathbb{Z}} \left(g * T_n\chi_{[-\frac{1}{2},\frac{1}{2}]}\right)(x) = \sum_{n\in\mathbb{Z}} (g * \varphi_n)(x) = \sum_{n\in\mathbb{Z}} \psi_n(x)\end{aligned}$$

for all $x \in \mathbb{R}$. For the boundedness of $\{\psi_n\}_{n\in\mathbb{Z}}$ in the space A we write

$$\|\psi_n\|_A = \left\|\overline{F}\psi_n\right\|_{W(C_0,\ell^1)} = \left\|\overline{F}(g * \varphi_n)\right\|_{W(C_0,\ell^1)} = \left\|\overline{F}g\overline{F}\varphi_n\right\|_{W(C_0,\ell^1)}, \tag{4.4}$$

where $\overline{F}$ denotes the inverse Fourier transform. Since $\psi_n = g * \varphi_n \in S(\mathbb{R}) \subset W\left(C_0, \ell^1\right)$ for all $n \in \mathbb{Z}$, then $\overline{F}(\psi_n) = \overline{F}(g * \varphi_n) = \overline{F}g\overline{F}\varphi_n \in S(\mathbb{R}) \subset W\left(C_0, \ell^1\right)$. Also

$$\overline{F}\varphi_n(x) = \overline{F}\left(T_n\chi_{[-\frac{1}{2},\frac{1}{2}]}\right)(x) = M_n\overline{F}\left(\chi_{[-\frac{1}{2},\frac{1}{2}]}\right), \tag{4.5}$$

where M_n is the modulation operator. By (4.4) and (4.5)

$$\|\psi_n\|_A = \left\|\overline{F}\psi_n\right\|_{W(C_0,\ell^1)} = \left\|\overline{F}g\overline{F}\varphi_n\right\|_{W(C_0,\ell^1)}$$

$$= \left\| \overline{F}gM_n\overline{F}\left(\chi_{[-\frac{1}{2},\frac{1}{2}]}\right)\right\|_{W(C_0,\ell^1)} = \left\|\overline{F}g\overline{F}\left(\chi_{[-\frac{1}{2},\frac{1}{2}]}\right)\right\|_{W(C_0,\ell^1)}$$
$$\leq \left\|\overline{F}g\right\|_{W(C_0,\ell^1)}\left\|\overline{F}\left(\chi_{[-\frac{1}{2},\frac{1}{2}]}\right)\right\|_{\infty} = C < \infty$$

for all $n\in\mathbb{Z}$. Hence

$$\sup_{n\in\mathbb{Z}}\|\psi_n\|_A = C < \infty, \tag{4.6}$$

which completes the proof. □

Let $B = \mathcal{F}L^{p(x)}(\mathbb{R})$. We know by Corollary 4.4 that $\mathcal{F}L^{p(x)}(\mathbb{R})$ is a homogeneous Banach space. By Proposition 4.1, $B \hookrightarrow S'(\mathbb{R})$. Hence A is a homoneous Banach space with respect to pointwise multiplication with the norm

$$\left\|\widehat{f}\right\|_A = \|f\|_{W(C_0,\ell^1)}.$$

Since $W\left(C_0,\ell^1\right) \subset L^1$, then $A = \mathcal{F}\left(W\left(C_0,\ell^1\right)\right) \subset \mathcal{F}L^1 \subset C_b(\mathbb{R})$. Define the unite map $I : A \to C_b(\mathbb{R})$. Then we write

$$\left\|I\widehat{f}\right\|_{L^\infty} = \left\|\widehat{f}\right\|_{L^\infty} \leq \|f\|_{L^1} \leq \|f\|_{W(C_0,\ell^1)} = \left\|\widehat{f}\right\|_A.$$

That means A is continuously embedded into $(C_b(\mathbb{R}), \|.\|_{L^\infty})$. Also since $S(\mathbb{R}) \subset W\left(C_0,\ell^1\right)$ then

$$\mathcal{F}(S(\mathbb{R})) = S(\mathbb{R}) \subset \mathcal{F}\left(W\left(C_0,\ell^1\right)\right) = A.$$

Now we are ready to define the Wiener amalgam space $W\left(\mathcal{F}L^{p(x)}, L^q_m\right)$ by [14].

Definition 4.7 Let the family of functions $\{\psi_i\}_{i\in I}$ on $\mathbb{R}$ be a BUPU. Also let $p(x)$ be variable exponents on $\mathbb{R}$, $1 \leq q \leq \infty$ and let m be a weight function $\mathbb{R}$. The Wiener amalgam space $W\left(\mathcal{F}L^{p(x)}, L^q_m\right)$ can be defined as follows: $f \in W\left(\mathcal{F}L^{p(x)}, L^q_m\right)$ if and only if $f \in S'(\mathbb{R})$ and

$$\|f\|_{W\left(\mathcal{F}L^{p(x)},L^q_m\right)} = \left(\sum_{n\in\mathbb{Z}}\|f\psi_n\|^q_{\mathcal{F}L^{p(x)}}\, m(n)^q\right)^{\frac{1}{q}} < \infty.$$

One can easily show as in [14] that these spaces do not depend on the bounded uniform partition of unity$\{\psi_n\}$.

Proposition 4.8 (i) The space $W\left(\mathcal{F}L^{p(x)}, L^q_m\right)$ is translation invariant and

$$\left\|T_a\widehat{f}\right\|_{W\left(\mathcal{F}L^{p(x)},L^q_m\right)} \leq m(a)\left\|\widehat{f}\right\|_{W\left(\mathcal{F}L^{p(x)},L^q_m\right)}$$

for all $\widehat{f} \in W\left(\mathcal{F}L^{p(x)}, L^q_m\right)$.

(ii) The translation operator $t \to T_t\widehat{f}$ is continuous from $\mathbb{R}$ into $W\left(\mathcal{F}L^{p(x)}, L^q_m\right)$ for all $\widehat{f} \in W\left(\mathcal{F}L^{p(x)}, L^q_m\right)$.

Proof (i) Let $\widehat{f} \in W\left(\mathcal{F}L^{p(x)}, L^q_m\right)$. Then $F_{\widehat{f}}(x) = \left\|\widehat{f}(t)\,\chi_{Q+x}(t)\right\|_{\mathcal{F}L^{p(x)}} \in L^q_m(\mathbb{R})$. Also we have

$$F_{T_a\widehat{f}}(x) = \left\|\left(T_a\widehat{f}\right)(t)\,\chi_{Q+x}(t)\right\|_{\mathcal{F}L^{p(x)}} = \left\|\widehat{f}(t)\,\chi_{Q+x-a}(t)\right\|_{\mathcal{F}L^{p(x)}}$$
$$= F_{\widehat{f}}(x-a) = T_aF_{\widehat{f}}(x).$$

Since $F_{\widehat{f}}(x) \in L^q_m(\mathbb{R})$ and $L^q_m(\mathbb{R})$ is translation invariant then $T_aF_{\widehat{f}}(x) = F_{T_a\widehat{f}}(x) \in L^q_m(\mathbb{R})$. We also obtain

$$\left\|T_a\widehat{f}\right\|_{W\left(\mathcal{F}L^{p(x)},L^q_m\right)} = \left\|F_{T_a\widehat{f}}\right\|_{L^q_m} = \left\{\int_{\mathbb{R}}\left|T_aF_{\widehat{f}}(x)\,m(x)\right|^q \mathrm{d}x\right\}^{\frac{1}{q}}$$

$$= \left\{ \int_{\mathbb{R}} \left| T_a F_{\widehat{f}} (u)\, m(u+a) \right|^q \mathrm{d}x \right\}^{\frac{1}{q}}$$
$$\leq m(a) \left\| F_{\widehat{f}} \right\|_{\mathcal{F}L^{p(x)}} = m(a) \|f\|_{W\left(\mathcal{F}L^{p(x)}, L^q_w\right)}.$$

(ii) Let $\widehat{f} \in W\left(\mathcal{F}L^{p(x)}, L^q_m\right)$. Since $W\left(\mathcal{F}L^{p(x)}, L^q_m\right)$ is translation invariant, for the proof of (ii) it is enough to show that the operator $t \to T_t \widehat{f}$ is continuous at $t = 0$. One can write

$$\left\| T_a \widehat{f} - \widehat{f} \right\|_{W\left(\mathcal{F}L^{p(x)}, L^q_m\right)} = \left\| F_{T_a \widehat{f}} - \widehat{f} \right\|_{L^q_m} \tag{4.7}$$

and

$$F_{T_a \widehat{f} - \widehat{f}}(x) = \left\| \left(T_a \widehat{f} - \widehat{f} \right)(t)\, \chi_{Q+x}(t) \right\|_{\mathcal{F}L^{p(x)}}$$
$$= \left\| (M_a f - f)^{\wedge}(t)\, \chi_{Q+x}(t) \right\|_{\mathcal{F}L^{p(x)}}$$
$$\leq \left\| (M_a f - f)^{\wedge} \right\|_{\mathcal{F}L^{p(x)}} = \|M_a f - f\|_{L^{p(x)}}.$$

By Lemma 5 in [3] , the map $a \to M_a f$ is continuous from $\mathbb{R}$ into $L^{p(x)}(\mathbb{R})$. Also we have

$$F_{T_a \widehat{f} - \widehat{f}}(x) \leq \|M_a f\|_{L^{p(x)}} + \|f\|_{L^{p(x)}} = 2\|f\|_{L^{p(x)}} < \infty.$$

Hence by Lebesgue dominated theorem the right hand side of (4.7) tends to zero as $a \to 0$. This completes the proof. □

Proposition 4.9 (i) $\left(W\left(\mathcal{F}L^{p(x)}, L^q_m\right), W\left(\mathcal{F}L^{\infty}, L^1_m\right), W\left(\mathcal{F}L^{p(x)}, L^q_m\right)\right)$ is a Banach convolution triples (BCT).

(ii) Let $\frac{1}{p(x)} + \frac{1}{r(x)} = \frac{1}{k(x)} \leq 1$. Then

$$\left(W\left(\mathcal{F}L^{p(x)}, L^q_m\right), W\left(\mathcal{F}L^{r(x)}, L^1_m\right), W\left(\mathcal{F}L^{k(x)}, L^q_m\right)\right)$$

is a BCT.

Proof (i) Since $\mathcal{F}L^{p(x)}$ is a Banach module over $\mathcal{F}L^{\infty}$ with respect to convolution Proposition 4.5, then $\left(\mathcal{F}L^{p(x)}, \mathcal{F}L^{\infty}, \mathcal{F}L^{p(x)}\right)$ is a Banach convolution triple. Also it is known that $\left(L^q_m, L^1_m, L^q_m\right)$ is a Banach convolution triple. Hence by Theorem 3 in [14], $(W(\mathcal{F}L^{p(x)}, L^q_m)$, $W(\mathcal{F}L^{\infty}, L^1_m), W(\mathcal{F}L^{p(x)}, L^q_m))$ is a Banach convolution triple.

(ii) Let $(F, G) \in \mathcal{F}L^{p(x)}(\mathbb{R}) \times \mathcal{F}L^{r(x)}(\mathbb{R})$. Then there exists $f \in L^{p(x)}(\mathbb{R})$ and $g \in L^{r(x)}(\mathbb{R})$ such that $\widehat{f} = F$ and $\hat{g} = G$. By Lemma 1.18 in [32], there exists a $C > 0$ such that

$$\|F * G\|_{\mathcal{F}L^{k(x)}} = \|fg\|_{L^{k(x)}} \leq C \|f\|_{L^{p(x)}} \|g\|_{L^{r(x)}}$$
$$= C \|F\|_{\mathcal{F}L^{p(x)}} \|G\|_{\mathcal{F}L^{r(x)}} < \infty$$

and $\mathcal{F}L^{p(x)}(\mathbb{R}) * \mathcal{F}L^{r(x)}(\mathbb{R}) \subset \mathcal{F}L^{k(x)}(\mathbb{R})$. Hence $\left(\mathcal{F}L^{p(x)}, \mathcal{F}L^{r(x)}, \mathcal{F}L^{k(x)}\right)$ is a BCT. Since $\left(L^q_m, L^1_m, L^q_m\right)$ is a BCT, then again by Theorem 3 in [14] the proof is complete. □

5 Boundedness of Hardy Littlewood Maximal Operator on $W\left(L^{p(x)}, L^q_w\right)$

Let $\Omega \subset \mathbb{R}$ be an open subset and let $\mathcal{P}(\Omega)$ be the set of measurable functions $p : \Omega \to [1, \infty)$ such that $1 < p_* \leq p(x) \leq p^* < \infty$. For $f \in L^1_{\text{loc}}(\Omega)$, we define the (centered) Hardy-Littlewood maximal function Mf of f by

$$Mf(x) = \sup_{r>0} \frac{1}{\left|\widetilde{B}(x,r)\right|} \int_{\widetilde{B}(x,r)} |f(y)|\,\mathrm{d}y, \quad \widetilde{B}(x,r) = B(x,r) \cap \Omega, \tag{5.1}$$

where the supremum is taken over all balls $\widetilde{B}(x,r)$ and $|\widetilde{B}(x,r)|$ denotes the volume of $\widetilde{B}(x,r)$.

It is known that the Hardy-Littlewood maximal function is not bounded in $L^{p(x)}(\Omega)$ in general [29].

We will often need to assume that $p(x)$ satisfies the following two log-Hölder continuity conditions:

$$|p(x) - p(y)| \le \frac{C}{-\ln|x-y|}, \ x, y \in \Omega, \ |x-y| \le \frac{1}{2}. \tag{5.2}$$

and

$$|p(x) - p(y)| \le \frac{C}{\ln(e + |x|)}, \ x, \ y \in \Omega, \ |y| > |x|. \tag{5.3}$$

We use the notation

$$\mathbf{P}(\Omega) := \left\{ p \in \mathcal{P}(\Omega) : M \text{ is bounded on } L^{p(x)}(\Omega) \right\}. \tag{5.4}$$

It is known that if Ω is open and bounded and (5.3) holds then the Hardy-Littlewood maximal operator is bounded on $L^{p(x)}(\Omega)$, [9]. It is also known that If Ω is open and both (5.3) and (5.4) hold then again the Hardy-Littlewood maximal operator is bounded on $L^{p(x)}(\Omega)$ (see [7], [8], [11], [22]). Although the Hardy-Littlewood maximal function is a bounded on $L^{p(x)}$ under some conditions, it is not bounded on many of the Wiener amalgam spaces. We know the following result.

Proposition 5.1 (Aydın, Gürkanlı, [4]) Let $p : \mathbb{R} \to [1, \infty)$, $1 \le q \le \infty$ and w is a weight function. If $\frac{1}{w} \in L^s(\mathbb{R})$ and $\frac{1}{q} + \frac{1}{s} = 1$, then the Hardy-Littlewood maximal function M is not bounded on $W\left(L^{p(x)}(\mathbb{R}), L^q_w(\mathbb{R})\right)$.

Now we will show that the Hardy-Littlewood maximal operator is bounded on $W\left(L^{p(x)}(\mathbb{R}), L^q_m(\mathbb{R})\right)$ under some conditions.

Theorem 5.2 Let $r(x) \le q \le p(x)$ and $\frac{1}{q} + \frac{1}{q'} = 1$. If $\frac{1}{m} \notin L^{q'}$ and the Hardy- Littlewood maximal operator

$$M : L^q_m(\mathbb{R}) \to L^q_m(\mathbb{R})$$

is bounded, then the Hardy -Littlewood maximal operator

$$M : W\left(L^{p(x)}, L^q_m\right) \to W\left(L^{r(x)}, L^q_m\right)$$

is bounded.

Proof By the continuity of

$$M : L^q_m(\mathbb{R}) \to L^q_m(\mathbb{R}), \tag{5.5}$$

there exists $C_1 > 0$ such that

$$\|Mf\|_{L^q_m} \le C_1 \|f\|_{L^q_m} \tag{5.6}$$

for all $f \in L^q_m(\mathbb{R})$. Also it is known by Proposition 11.5.2 in [23] that $W(L^q, L^q_m) = L^q_m(\mathbb{R})$. Since $r(x) \le q \le p(x)$, then by Proposition 2.5 in [4], we have the embeddings

$$W\left(L^{p(x)}, L^q_m\right) \hookrightarrow W(L^q, L^q_m) = L^q_m(\mathbb{R}) \hookrightarrow W\left(L^{r(x)}, L^q_m\right). \tag{5.7}$$

Hence the unit maps

$$I_{W\left(L^{p(x)},L^q_m\right)} : W\left(L^{p(x)}, L^q_m\right) \to L^q_m\left(\mathbb{R}\right)$$

and

$$I_{L^q_m(\mathbb{R})} : L^q_m\left(\mathbb{R}\right) \to W\left(L^{r(x)}, L^q_m\right)$$

are bounded. Then there exist $C_2 > 0$ and $C_3 > 0$ such that

$$\|f\|_{L^q_m} \leq C_2 \|f\|_{W\left(L^{p(x)},L^q_m\right)} \tag{5.8}$$

and

$$\|g\|_{W\left(L^{r(x)},L^q_m\right)} \leq C_3 \|g\|_{L^q_m} \tag{5.9}$$

for all $f \in W\left(L^{p(x)}, L^q_m\right)$ and $g \in L^q_m\left(\mathbb{R}^n\right)$. Now let $f \in W\left(L^{p(x)}, L^q_m\right)$ be given. By the embeddings (5.7) and the continuity of the Hardy-Littlewood maximal operator $M : L^q_m\left(\mathbb{R}\right) \to L^q_m\left(\mathbb{R}\right)$ we have $f \in L^q_m\left(\mathbb{R}\right)$ and $Mf \in L^q_m\left(\mathbb{R}\right)$. Hence by (5.6), (5.8) and (5.9) we obtain

$$\begin{aligned}\|Mf\|_{W\left(L^{r(x)},L^q_m\right)} &= \left\|\left(I_{L^q_m(\mathbb{R})} \circ M\right) f\right\|_{W\left(L^{r(x)},L^q_m\right)} = \left\|I_{L^q_m(\mathbb{R})}\left(Mf\right)\right\|_{W\left(L^{r(x)},L^q_m\right)} \\ &\leq C_3 \|Mf\|_{L^q_m} \leq C_1C_3 \|f\|_{L^q_m} \leq C_1C_2C_3 \|f\|_{W\left(L^{p(x)},L^q_m\right)} \\ &= K \|f\|_{W\left(L^{p(x)},L^q_m\right)},\end{aligned}$$

where $K = C_1C_2C_3$. □

The following Corollary can be obtained by Theorem 5.2 and Theorem 2.2 in [10].

Corollary 5.3 Let $p(x)$ and q be as in Theorem 5.2 and let $1 \leq s < \infty$. Then $s.p(x) \in \mathbf{P}\left(\mathbb{R}\right)$ and Hardy-Littlewood maximal operator

$$M : W\left(L^{s.p(x)}, L^q_m\right) \to W\left(L^{p(x)}, L^q_m\right)$$

is bounded.

One can easily prove the following theorem by using Proposition 2.5 in [4], Theorem 1.5 in [8] and the same technic in the proof of Theorem 5.2.

Proposition 5.4 Let $p \in \mathbf{P}\left(\mathbb{R}\right)$.

i) If $L^{p(x)} \hookrightarrow W\left(L^{r_1(x)}, L^{q_1}_m\right))$, $r\left(x\right) \leq r_1\left(x\right)$, $q \geq q_1 > 1$ and $\frac{1}{m} \notin L^{q'}$, then the Hardy-Littlewood maximal operator

$$M : L^{p(x)}\left(\mathbb{R}\right) \to W\left(L^{r(x)}, L^q_m\right)$$

is bounded, where $\frac{1}{q} + \frac{1}{q'} = 1$.

ii) If $W\left(L^{r_2(x)}, L^{q_2}_m\right) \hookrightarrow L^{p(x)}$, $q_2 > 1$ and $\frac{1}{m} \notin L^{q_2'}$, then the Hardy-Littlewood maximal operator

$$M : W\left(L^{r_2(x)}, L^{q_2}_m\right) \to L^{p(x)}\left(\mathbb{R}\right)$$

is bounded.

The following Corollary is proved easily by Proposition 5.4.

Corollary 5.5 Let the exponents be as in Proposition 5.4. If $L^{p(x)} \hookrightarrow W\left(L^{r_1(x)}, L^{q_1}_m\right)$ and $W\left(L^{r_2(x)}, L^{q_2}_m\right) \hookrightarrow L^{p(x)}$ then the Hardy-Littlewood maximal operator

$$M : W\left(L^{r_2(x)}, L^{q_2}_m\right) \to W\left(L^{r(x)}, L^q_m\right)$$

is bounded.

Example 5.6 Let m be a weight function satisfying $\frac{1}{m} \notin L^2(\mathbb{R})$. Assume that $p, r : \mathbb{R} \to [1, \infty)$ are functions defining by

$$p(x) = \left\{ \begin{array}{ll} 2, & \text{for } x < 0 \\ 4, & \text{for } 0 \leqslant x \leqq 1 \\ 2, & \text{for } x > 1 \end{array} \right\}, \quad r(x) = \left\{ \begin{array}{ll} 1, & \text{for } x < 0 \\ 2, & \text{for } 0 \leqslant x \leqq 1 \\ 1, & \text{for } x > 1 \end{array} \right\}.$$

Since $r(x) < 2 < p(x)$ then by Theorem 5.2, the Hardy-Littlewood maximal operator

$$M : W\left(L^{p(x)}, L^q_m\right) \to W\left(L^{r(x)}, L^q_m\right)$$

is bounded.

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