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A class of second order nondifferentiable symmetric duality relations under generalized assumptions

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Abstract

In this article, a pair of second-order nondifferentiable symmetric dual model in optimization problem is formulated over arbitrary cones. For a differentiable function, we consider the definition of strongly K-pseudobonvexity convexity. Next, we derive the appropriate duality results under aforesaid assumptions.

Keywords: Symmetric duality, second-order, non-differentiable, strongly K-pseudobonvexity.

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1. Introduction

In the recent years, the analysis of multiobjective optimization problems has been a focal issue. Such multiobjective optimization problems are useful mathematical models for the investigation of real-world problems. Also many researchers motivated by works of the concept of second-order duality in nonlinear problems, introduced by Mangasarian [10] have also worked in this area. The advantage of second-order duality is considered over first-order as it gives more closer bounds. Hanson [8] in his study has cited one example which demonstrates the application of second-order duality in somewhat different perspective.

In the last several years, several optimality and duality results have been obtained for multiobjective fractional programming problems. Multiobjective fractional problem and its duality relations have been assumed under higher-order (f, α, ρ, d)-convexity by Chen in [1]. Later on in [13], Suneja et al. derived the higher-order Mond-Weir and Schaible type nondifferentiable dual programs and their duality theorems under higher-order (f, ρ, σ)-type I-assumptions. Ying [15] studied higher-order multiobjective symmetric fractional problem and formulated its Mond-Weir type dual and duality theorems were proved under

higher-order (f, α, ρ, d)-convexity assumptions. Many researchers have also worked in the same directions (see [3–6, 9, 11, 14]).

The purpose of the present work is to study second order symmetric duality over arbitrary cones for nondifferentiable type programs under strongly K-pseudobonvexity assumptions. The paper is organized as follows. In Section 2, we present some relevant and basic preliminaries. In Section 3, we formulate a nondifferentiable second order symmetric dual problem with cone constraints and derive appropriate duality results under strongly K-pseodobonvexity assumptions.

2. Preliminaries and definitions

Definition 2.1 ([2]). Let C be a compact convex set in R^n. The support function of C is defined by

$$s(x|C) = \max\{x^T y : y \in C\}.$$

A support function, being convex and everywhere finite, has a subdifferential, that is, there exists a $z \in R^n$ such that $s(y|C) \geqslant s(x|C) + z^T(y - x)$ for every $x \in C$. The subdifferential of $s(x|C)$ is given by

$$\partial s(x|C) = \{z \in C : z^T x = s(x|C)\}.$$

For a convex set $D \subset R^n$, the normal cone to D at a point $x \in D$ is defined by

$$N_D(x) = \{y \in R^n : y^T(z - x) \leqq 0, \text{for all } z \in D\}.$$

When C is a compact convex set, $y \in N_C(x)$ if and only if $s(y|C) = x^T y$, or equivalently, $x \in \partial s(y|C)$.

Let $X \subset R^n$ and $K \subset R^k$ be a closed convex pointed cone with $\text{int}K \neq \phi$. Now, we consider the definitions of K-pseudoinvex and K-pseudobonvex functions with respect to η.

Definition 2.2. A differentiable function $f : X \to R^k$ is K-pseudoinvex at $u \in X$ with respect to $\eta : X \times X \longrightarrow R^n$, if for every $(x, p) \in X \times R^n$,

$$-\eta^T(x, u)[\nabla_u f(u)] \notin \text{int}(K) \Longrightarrow -[f(x) - f(u)] \notin \text{int}(K).$$

Definition 2.3. A differentiable function $f : X \to R^k$ is strongly K-pseudoinvex at $u \in X$ with respect to $\eta : X \times X \longrightarrow R^n$, if for every $(x, p) \in X \times R^n$,

$$-\eta^T(x, u)[\nabla_u f(u)] \notin \text{int}(K) \Longrightarrow [f(x) - f(u)] \in K.$$

Definition 2.4. A differentiable function $f : X \to R^k$ is K-pseudobonvex at $u \in X$ with respect to $\eta : X \times X \longrightarrow R^n$, if for every $(x, p) \in X \times R^n$,

$$-\eta^T(x, u)[\nabla_u f(u) + \nabla_{uu} f(u)\}p] \notin \text{int}(K) \Longrightarrow -[f(x) - f(u) + \frac{1}{2}p^T \nabla_{uu} f(u)p] \notin \text{int}(K).$$

Definition 2.5. A differentiable function $f : X \to R^k$ is strongly K-pseudobonvex at $u \in X$ with respect to $\eta : X \times X \longrightarrow R^n$, if for every $(x, p) \in X \times R^n$,

$$-\eta^T(x, u)[\nabla_u f(u) + \nabla_{uu} f(u)p] \notin \text{int}(K) \Longrightarrow [f(x) - f(u) + \frac{1}{2}p^T \nabla_{uu} f(u)p] \in K.$$

Remark 2.6. $\frac{1}{2}p^T \nabla_{uu} f(u)p$ represents the vector of components

$$\left(\frac{1}{2}p^T \nabla_{uu} f_1(u)p, \frac{1}{2}p^T \nabla_{uu} f_2(u)p, \ldots, \frac{1}{2}p^T \nabla_{uu} f_k(u)p \right).$$

Remark 2.7. For proving the strong and converse duality theorems, we need necessary conditions proposed by Suneja et al. in [12]. They considered the the following multiobjective programming problem:

$$\textbf{(VP)} \quad \text{K-Minimize } f(x),$$
$$\text{subject to } X_0 = \{x \in R^n : \ -g(x) \in Q, \ x \in C\},$$

where $f : R^n \to R^k$, $g : R^n \to R^m$, $C \subset R^n$, K and Q are closed convex cones with nonempty interiors in R^k and R^m, respectively.

Definition 2.8. A point $u \in X_0$ is said to be a weak minimum of (VP), if there exists no other $x \in X_0$ such that $f(u) - f(x) \in \text{int}(K)$.

Definition 2.9. The positive polar cone K^* of K is defined as follows:

$$K^* = \{z \in R^p : x^T z \geqslant 0, \text{for all } x \in K\}.$$

Lemma 2.10. *If $u \in X_0$ is a weak minimum of (VP), then there exists $\bar{\alpha} \in K^*$ and $\beta \in Q^*$ not both zero such that*

$$(\bar{\alpha}^T \nabla f(u) + \bar{\beta}^T \nabla g(u))^T (x - u) \geqslant 0, \quad \forall \, x \in C,$$
$$\bar{\beta}^T \nabla g(u) = 0.$$

3. Second-order nondifferentiable symmetric duality model

In this section, we formulate the following pair of second-order nondifferentiable symmetric dual programs over arbitrary constraints:

$$\textbf{(NSOP)} \text{K-Minimize } W(x, y, r, p) = f(x, y) + s(x|B) - y^T r - \frac{1}{2} p^T \nabla_{yy} f(x, y) p,$$

$$\text{subject to} \quad -\sum_{i=1}^{k} \lambda_i \left[\nabla_y f_i(x, y) - r_i + \nabla_{yy} f_i(x, y)\} p \right] \in C_2^*,$$

$$y^T \left[\sum_{i=1}^{k} \lambda_i \left(\nabla_y f_i(x, y) - r_i + \nabla_{yy} f_i(x, y) p \right) \right] \geqslant 0,$$

$$p^T \left[\sum_{i=1}^{k} \lambda_i \left(\nabla_y f_i(x, y) - r_i + \nabla_{yy} f_i(x, y) p \right) \right] \geqslant 0,$$

$$x \in C_1, \ r \in F, \ \lambda \in \text{int}(K^*),$$

$$\textbf{(NSOD)} \text{ K-Maximize } T(u, v, t, q) = f(u, v) - s(v|F) + u^T t - \frac{1}{2} q^T \nabla_{uu} f(u, v) q,$$

$$\text{subject to} \quad \sum_{i=1}^{k} \lambda_i \left[\nabla_u f_i(u, v) + t_i + \nabla_{uu} f_i(u, v) q \right] \in C_1^*, \tag{3.1}$$

$$u^T \left[\sum_{i=1}^{k} \lambda_i \left(\nabla_u f_i(u, v) + t + \nabla_{uu} f_i(u, v) q \right) \right] \leqslant 0, \tag{3.2}$$

$$q^T \left[\sum_{i=1}^{k} \lambda_i \left(\nabla_u f_i(u, v) + t + \nabla_{uu} f_i(u, v) q \right) \right] \leqslant 0, \tag{3.3}$$

$$v \in C_2, \ t \in B, \ \lambda \in \text{int}(K^*),$$

where C_1 and C_2 are closed convex cones with non-empty interiors in R^n and R^m, respectively. C_1^* and C_2^* are positive polar cones of C_1 and C_2, respectively, and $f : R^n \times R^m \to R^k$ is differentiable function.

Let R^0 and S^0 be feasible solution of (NSOP) and (NSOD), respectively.

Theorem 3.1 (Weak duality theorem). *Let* $(x, y, \lambda, r, p) \in R^0$ *and* $(u, v, \lambda, t, q) \in S^0$. *Let*

  (i) $f(., v)$ *be strongly K-pseudobonvex and* $(.)^{\mathsf{T}}t$ *be strongly K- psedoinvex at* u *with respect to* η;

 (ii) $-f(x, .)$ *be K-pseudobonvex and* $(.)^{\mathsf{T}}r$ *be K- psedoinvex at* y *with respect to* ξ;

(iii) $\eta(x, u) + u + q \in C_1$ *for all* $x \in C_1$;

(iv) $\xi(v, y) + y + p \in C_2$ *for all* $y \in C_2$.

Then

$$T(u, v, \lambda, t, q) - W(x, y, \lambda, r, p) \notin int(K).$$

Proof. Suppose, on contrary

$$T(u, v, \lambda, t, q) - W(x, y, \lambda, r, p) \in int(K),$$

i.e.,

$$\left\{ f(u, v) - s(v|F) + u^{\mathsf{T}}t - \frac{1}{2}q^{\mathsf{T}}\nabla_{uu}f(u, v)q \right.$$
$$\left. - \left(f(x, y) + s(x|B) - y^{\mathsf{T}}r - \frac{1}{2}p^{\mathsf{T}}\nabla_{yy}f(x, y)p \right) \right\} \in int(K). \tag{3.4}$$

From, hypothesis (iii) and (3.1), we obtain

$$(\eta(x, u) + u + q)^{\mathsf{T}} \left[\sum_{i=1}^{k} \lambda_i \left(\nabla_u f_i(u, v) + t_i + \nabla_{uu}f_i(u, v)q \right) \right] \geqslant 0,$$

$$\eta^{\mathsf{T}}(x, u) \left[\sum_{i=1}^{k} \lambda_i \left(\nabla_u f_i(u, v) + t_i + \nabla_{uu}f_i(u, v)q \right) \right] \geqslant -(u + q) \left[\sum_{i=1}^{k} \lambda_i \left(\nabla_u f_i(u, v) + t_i + \nabla_{uu}f_i(u, v)q \right) \right],$$

which on using (3.2) and (3.3), it yields

$$\eta^{\mathsf{T}}(x, u) \left[\sum_{i=1}^{k} \lambda_i \left(\nabla_u f_i(u, v) + t_i + \nabla_{uu}f_i(u, v)q \right) \right] \geqslant 0$$

or

$$\sum_{i=1}^{k} \lambda_i \left(\eta^{\mathsf{T}}(x, u) \left[\nabla_u f_i(u, v) + t_i + \nabla_{uu}f_i(u, v)q \right] \right) \geqslant 0. \tag{3.5}$$

Since $\lambda \in int\ (K^*)$, we have

$$-\eta^{\mathsf{T}}(x, u) \left[\nabla_u f(u, v) + t + \nabla_{uu}f(u, v)q \right] \notin int(K). \tag{3.6}$$

From hypothesis (i), we obtain

$$\left[f(x, v) - f(u, v) + \frac{1}{2}q^{\mathsf{T}}\nabla_{uu}f(u, v)q \right] \in K$$

and

$$[x^{\mathsf{T}}t - u^{\mathsf{T}}t] \in K.$$

Combining above inequalities, we get

$$[f(x, v) + x^{\mathsf{T}}t - f(u, v) - u^{\mathsf{T}}t + \frac{1}{2}q^{\mathsf{T}}\nabla_{uu}f(u, v)q] \in K. \tag{3.7}$$

From (3.6) and (3.7), we get

$$[f(x, v) + x^\mathsf{T} t - f(u, v) - u^\mathsf{T} t + \frac{1}{2} q^\mathsf{T} \nabla_{uu} f(u, v) q] \in K. \tag{3.8}$$

Similarly, using hypothesis (iii) and primal constraints, it follows that

$$\xi^\mathsf{T}(v, y) [\nabla_y f(x, y) - r + \nabla_{yy} f(x, y) p] \notin \text{int}(K).$$

Again, from hypothesis (ii), we obtain

$$[-f(x, y) + y^\mathsf{T} r + f(x, v) - v^\mathsf{T} r + \frac{1}{2} p^\mathsf{T} \nabla_{yy} f(x, y) p] \notin \text{int}(K). \tag{3.9}$$

Adding inequalities(3.4)-(3.8) and using the fact that $x^\mathsf{T} t \leqslant s(x|B)$ and $v^\mathsf{T} r \leqslant s(v|F)$, we have

$$[-f(x, y) + y^\mathsf{T} r + f(x, v) - v^\mathsf{T} r + \frac{1}{2} p^\mathsf{T} \nabla_{yy} f(x, y) p] \in \text{int}(K),$$

which contradicts (3.9). Hence the result is obtained.   □

Theorem 3.2 (Weak duality theorem). *Let* $(x, y, \lambda, r, p) \in R^0$ *and* $(u, v, \lambda, t, q) \in S^0$. *Let*

(i) $f(., v)$ *be* K-*pseudobonvex and* $(.)^\mathsf{T} t$ *be* K-*psedoinvex at* u *with respect to* η;
(ii) $f(x, .)$ *be strongly* K-*pseudoconcave and* $(.)^\mathsf{T} r$ *be strongly* K- *psedoincave at* y *with respect to* ξ;
(iii) $\eta(x, u) + u + q \in C_1$ *for all* $x \in C_1$;
(iv) $\xi(v, y) + y + p \in C_2$ *for all* $y \in C_2$.

Then

$$T(u, v, t, q) - W(x, y, r, p) \notin \text{int}(K).$$

Proof. The proof is clear from Theorem 3.1.   □

For proving the strong and converse duality results, it is convenient to rewrite the (NSOP) and (NSOD) in the given form.

(NSOP)   K-Minimize $W(x, y, r, p) = \Big(f_1(x, y) + s(x|B_1) - y^\mathsf{T} r_1 - \frac{1}{2} p^\mathsf{T} \nabla_{yy} f_1(x, y) p, \; f_2(x, y)$

$$+ s(x|B_2) - y^\mathsf{T} r_2 - \frac{1}{2} p^\mathsf{T} \nabla_{yy} f_2(x, y) p, \ldots, f_k(x, y) + s(x|B_k)$$

$$- y^\mathsf{T} r_k - \frac{1}{2} p^\mathsf{T} \nabla_{yy} f_k(x, y) p \Big),$$

subject to $-\sum_{i=1}^{k} \lambda_i \Big[\nabla_y f_i(x, y) - r_i + \nabla_{yy} f_i(x, y) \} p \Big] \in C_2^*,$

$$y^\mathsf{T} \Big[\sum_{i=1}^{k} \lambda_i \Big(\nabla_y f_i(x, y) - r_i + \nabla_{yy} f_i(x, y) p \Big) \Big] \geqslant 0,$$

$$p^\mathsf{T} \Big[\sum_{i=1}^{k} \lambda_i \Big(\nabla_y f_i(x, y) - r_i + \nabla_{yy} f_i(x, y) p \Big) \Big] \geqslant 0,$$

$$x \in C_1, \; r_i \in F_i \; (i = 1, 2, \ldots, k), \; \lambda \in \text{int}(K^*),$$

(NSOD)   K-Maximize $T(u, v, t, q) = \Big(f_1(u, v) - s(v|F_1) + u^\mathsf{T} t_1 - \frac{1}{2} q^\mathsf{T} \nabla_{uu} f_1(u, v) q, \; f_2(u, v)$

$$-s(\nu|F_2) + u^T t_2 - \frac{1}{2} q^T \nabla_{uu} f_2(u,\nu) q, \ldots, f_k(u,\nu) - s(\nu|F_k)$$

$$+ u^T t_k - \frac{1}{2} q^T \nabla_{uu} f_k(u,\nu) q \bigg),$$

subject to
$$\sum_{i=1}^{k} \lambda_i \left[\nabla_u f_i(u,\nu) + t_i + \nabla_{uu} f_i(u,\nu) q \right] \in C_1^*,$$

$$u^T \left[\sum_{i=1}^{k} \lambda_i \left(\nabla_u f_i(u,\nu) + t_i + \nabla_{uu} f_i(u,\nu) q \right) \right] \leqslant 0,$$

$$q^T \left[\sum_{i=1}^{k} \lambda_i \left(\nabla_u f_i(u,\nu) + t_i + \nabla_{uu} f_i(u,\nu) q \right) \right] \leqslant 0,$$

$$\nu \in C_2, \ t_i \in B_i \ \ (i=1,2,\ldots,k), \ \lambda \in \text{int}(K^*).$$

Theorem 3.3 (Strong duality theorem). *Let $(\bar{x}, \bar{y}, \bar{r}, \bar{p})$ be a weak minimum of (NSOP), and fix $\lambda = \bar{\lambda}$ in (NSOD). Assume that*

(i) *for every $(\nu, q) \in X \times R^m$, $\nu \in C_2$, $\{\nabla_u(f_i \bar{u}, \bar{\nu}) + \bar{t}_i + \nabla_{uu} f_i(\bar{u}, \bar{\nu}) \bar{q}\}$ is linearly independent, for every $i = 1, 2, \ldots, k$;*
(ii) *$\{\nabla_{uu} f_i(\bar{u}, \bar{\nu})\}$ is nonsingular, for every $i = 1, 2, \ldots, k$;*
(iii) *if $\bar{q}^T[\nabla_u f_i(\bar{u}, \bar{\nu}) + \bar{t}_i + \nabla_{uu} f_i(\bar{u}, \bar{\nu}) \bar{q}]] = 0$, then $\bar{q} = 0$;*
(iv) *$R_+^k \subset K$.*

Then, there exists $\bar{r} \in F$ such that $(\bar{x}, \bar{y}, \bar{r}, \bar{p})$ is a weak minimum for the problem (NSOD).

Proof. The proof of strong duality theorems follows almost same lines given by [7]. □

Remark 3.4. In case of symmetric programming problem, the proof remains the same as strong duality theorems.

Theorem 3.5 (Strict converse duality theorem). *Let $(\bar{u}, \bar{v}, \bar{t}, \bar{q})$ be a weak minimum of (NSOD), and fix $\lambda = \bar{\lambda}$ in (NSOP). Assume that*

(i) *for every $(x, p) \in X \times R^m$, $y \in C_2$, $\{\nabla_y(f_i \bar{x}, \bar{y}) - \bar{r}_i + \nabla_{yy} f_i(\bar{x}, \bar{y}) \bar{p}\}$ is linearly independent for every $i = 1, 2, \ldots, k$;*
(ii) *$\{\nabla_{yy} f_i(\bar{x}, \bar{y})\}$ is nonsingular, for every $i = 1, 2, \ldots, k$;*
(iii) *if $\bar{p}^T[\nabla_y f_i(\bar{x}, \bar{y}) - \bar{r}_i + \nabla_{yy} f_i(\bar{x}, \bar{y}) \bar{p}]] = 0$, then $\bar{p} = 0$;*
(iv) *$R_+^k \subset K$.*

Then, there exists $\bar{t} \in B$ such that $(\bar{u}, \bar{v}, \bar{t}, \bar{q})$ is a weak minimum for the problem (NSOD).

Proof. The proof is obvious from Theorem 3.3. □

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