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Inclusions and the approximate identities of the generalized grand Lebesgue spaces

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Abstract: Let $(\Omega, \sum, \mu)$ and $(\Omega, \sum, v)$ be two finite measure spaces and let $L^{p),\theta}(\mu)$ and $L^{q),\theta}(v)$ be two generalized grand Lebesgue spaces $[9, 10]$, where $1 < p, q < \infty$ and $\theta \geq 0$. In Section 2 we discuss the inclusion properties of these spaces and investigate under what conditions $L^{p),\theta}(\mu) \subseteq L^{q),\theta}(v)$ for two different measures μ and v. Let Ω be a bounded subset of $\mathbb{R}^n$. We know that the Lebesgue space $L^p(\mu)$ admits an approximate identity, bounded in $L^1(\mu), [5, 8]$. In Section 3 we investigate the approximate identities of $L^{p),\theta}(\mu)$ and show that it does not admit such an approximate identity. Later we discuss aproximate identities of the space $[L^p]_{p),\theta}$, the closure of $C_c^\infty(\Omega)$ in $L^{p),\theta}(\mu)$, where $C_c^\infty(\Omega)$ denotes the space of infinitely differentiable complex-valued functions with compact support on $\mathbb{R}^n$.

Key words: Lebesgue space, grand Lebesgue space, generalized grand Lebesgue space

1. Introduction

Let $(\Omega, \sum, \mu)$ be a measure space. It is well known that $\ell^p(\Omega) \subseteq \ell^q(\Omega)$ whenever $0 < p \leq q \leq \infty$. Subramanian [19] investigated all positive measures μ on Ω for which $L^p(\mu) \subseteq L^q(\mu)$ whenever $0 < p \leq q \leq \infty$. Romero [17] improved and completed some results of Subramanian. Miamee [13] considered the more general inclusion $L^p(\mu) \subseteq L^q(v)$, where μ and v are two measures. Gürkanlı [10] generalized these results to the Lorentz spaces.

Let Ω be a nonempty set, $\sum$ a σ-algebra of subsets of Ω and μ a positive finite measure on the measurable space $(\Omega, \sum)$. The grand Lebesgue space $L^{p)}(\mu)$ was introduced in $[11]$. This is a Banach space defined by the norm

$$\|f\|_{p)} = \sup_{0<\varepsilon\leq p-1} \left(\varepsilon \int_\Omega |f|^{p-\varepsilon} d\mu \right)^{\frac{1}{p-\varepsilon}};$$

where $1 < p < \infty$. For $0 < \varepsilon \leq p-1$, $L^p(\mu) \subset L^{p)}(\mu) \subset L^{p-\varepsilon}(\mu)$ hold. For some properties and applications of $L^{p)}(\mu)$ spaces we refer to papers $[1-4, 6, 11]$. A generalization of the grand Lebesgue spaces are the spaces

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$L^{p),\theta}(\mu)$, $\theta \geq 0$, defined by the norm (see [1, 11])

$$\|f\|_{p),\theta,\mu} = \|f\|_{p),\theta} = \sup_{0<\varepsilon\leq p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left(\int_{\Omega} |f|^{p-\varepsilon} d\mu \right)^{\frac{1}{p-\varepsilon}} = \sup_{0<\varepsilon\leq p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{p-\varepsilon} < \infty;$$

when $\theta = 0$ the space $L^{p),0}(\mu)$ reduces to the Lebesgue space $L^p(\mu)$ and when $\theta = 1$ the space $L^{p),1}(\mu)$ reduces to the grand Lebesgue space $L^{p)}(\mu)$. More precisely, we have for all $1 < p < \infty$ and $0 < \varepsilon \leq p - 1$

$$L^p(\mu) \subset L^{p),\theta}(\mu) \subset L^{p-\varepsilon}(\mu).$$

Different properties and applications of these spaces were discussed in [1, 2, 6, 7, 9].

If μ and υ are two measures on a $\sigma-$algebra $\sum$ of subsets of Ω, we say that υ is absolutely continuous with respect to μ if $\upsilon(E) = 0$ for every $E \in \sum$ such that $\mu(E) = 0$. We denote it by the symbol $\upsilon \ll \mu$. If μ and υ are absolutely continuous with respect to each other (i.e $\upsilon \ll \mu$ and $\mu \ll \upsilon$) then we denote it by the symbol $\mu \approx \upsilon$.

Let A be a Banach algebra. A Banach space $(B, \|.\|_B)$ is called Banach module over $(A, \|.\|_A)$ if B is a module over A in the algebraic sense for some multiplication, $(a, b) \to a.b$, and satisfies

$$\|a.b\|_B \leq \|a\|_A \|b\|_B.$$

An approximate identity in a Banach algebra A is a net $(e_\alpha)_{\alpha\in I} \subset A$ such that for every $f \in A$,

$$\lim_{\alpha} \|fe_\alpha - f\| = 0.$$

For two Banach modules B_1 and B_2 over a Banach algebra A, we write $M_A(B_1, B_2)$ or $Hom_A(B_1, B_2)$ for the space of all bounded linear operators T from B_1 into B_2 satisfying $T(ab) = aT(b)$ for all $a \in A, b \in B_1$. These operators are called multipliers (right) or module homomorphism from B_1 into B_2, [12, 14 − 16]. By Corollary 2.13 in [15],

$$Hom_A(B_1, B_2^*) \cong (B_1 \otimes_A B_2)^*,$$

where B_2^* is the dual of B and $\otimes_A$ is the $A-$ module tensor product.

2. Inclusions of generalized grand Lebesgue spaces

In this section we will accept that $1 < p, q < \infty$, $\theta \geq 0$, and $(\Omega, \sum)$ is a measurable space and all measures are defined on the $\sigma-$algebra $\sum$.

Lemma 1 *Let $(\Omega, \sum, \mu)$ and $(\Omega, \sum, \upsilon)$ be two finite measure spaces. Then the inclusion $L^{p),\theta}(\mu) \subseteq L^{q),\theta}(\upsilon)$ holds in the sense of equivalence classes if and only if μ and υ are absolutely continuous with respect to each other (i.e $\mu \approx \upsilon$) and $L^{p),\theta}(\mu) \subseteq L^{q),\theta}(\upsilon)$ in the sense of individual functions.*

Proof Suppose that $L^{p),\theta}(\mu) \subseteq L^{q),\theta}(\upsilon)$ in the sense of equivalence classes. Let $f \in L^{p),\theta}(\mu)$ be any individual function. Then $f \in L^{p),\theta}(\mu)$ in the sense of equivalence classes. By assumption, $f \in L^{q),\theta}(\upsilon)$ in the sense of equivalence classes. This implies $f \in L^{q),\theta}(\upsilon)$ in the sense of individual functions. Then $L^{p),\theta}(\mu) \subseteq L^{q),\theta}(\upsilon)$

in the sense of individual functions. To show $\upsilon \ll \mu$, take any set $E \in \sum$ with $\mu(E) = 0$. Then $\chi_E = 0$, $\mu - a.e$, and it is in the equivalence classes of $0 \in L^p(\mu)$, where χ_E is the characteristic function of E. By the inclusion $L^p(\mu) \subseteq L^{p),\theta}(\mu) \subseteq L^{q),\theta}(\upsilon)$ in the sense of equivalence classes, we have $0 \in L^{q),\theta}(\upsilon)$. Then

$$\sup_{0<\varepsilon\leq q-1} \varepsilon^{\frac{\theta}{q-\varepsilon}} \left[\upsilon(E)\right]^{\frac{1}{q-\varepsilon}} = \sup_{0<\varepsilon\leq q-1} \varepsilon^{\frac{\theta}{q-\varepsilon}} \left\|\chi_E\right\|_{q-\varepsilon} = \left\|\chi_E\right\|_{q),\theta} = 0. \tag{1}$$

Since $L^{q),\theta}(\upsilon) \subset L^{q-\varepsilon}(\upsilon)$, there exists a constant $C > 0$ such that

$$\left\|\chi_E\right\|_{p-\varepsilon} \leq C \left\|\chi_E\right\|_{q),\theta}.$$

Then by (1) we have $\chi_E = 0$, $\upsilon - a.e.$ Thus, $\upsilon(E) = 0$ and so $\upsilon \ll \mu$. Similarly, one can prove that $\mu \ll \upsilon$. The proof of the other direction is clear. □

Theorem 1 *Let $(\Omega, \sum, \mu)$ and $(\Omega, \sum, \upsilon)$ be two finite measure spaces. Then $L^{p),\theta}(\mu) \subseteq L^{q),\theta}(\upsilon)$ holds in the sense of equivalence classes if and only if $\mu \approx \upsilon$ and there exists a constant $C(p,q) > 0$ such that*

$$\|f\|_{q),\theta,\upsilon} \leq C(p,q) \|f\|_{p),\theta,\mu} \tag{2}$$

for all $f \in L^{p),\theta}(\mu)$.

Proof Assume that the inequality (2) is satisfied and $\mu \approx \upsilon$. By the inequality (2) the inclusion $L^{p),\theta}(\mu) \subseteq L^{q),\theta}(\upsilon)$ holds in the sense of individual functions. Then by Lemma 1, the inclusion $L^{p),\theta}(\mu) \subseteq L^{q),\theta}(\upsilon)$ holds in the sense of equivalence classes.

Conversely, assume that $L^{p),\theta}(\mu) \subseteq L^{q),\theta}(\upsilon)$ holds in the sense of equivalence classes. The grand Lebesgue space $L^{p),\theta}(\mu)$ is a Banach space with the sum norm

$$\|f\| = \|f\|_{p),\theta,\mu} + \|f\|_{q),\theta,\upsilon}.$$

Indeed, if we get any Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in the normed space $\left(L^{p),\theta}(\mu), \|.\|\right)$, it is also a Cauchy sequence in the spaces $\left(L^{p),\theta}(\mu), \|.\|_{p),\theta,\mu}\right)$ and $\left(L^{q),\theta}(\upsilon), \|.\|_{q),\theta,\upsilon}\right)$. Then $(f_n)_{n\in\mathbb{N}}$ converges to functions f and g in spaces $L^{p),\theta}(\mu)$ and $L^{q),\theta}(\upsilon)$, respectively. Thus, one can find a subsequence (f_{n_i}) of (f_n) such that $f_{n_i} \to f$, $\mu - a.e$ and $f_{n_i} \to g$, $\upsilon - a.e.$ Since υ is absolutely continuous with respect to μ, then $f_{n_i} \to f$, $\upsilon - a.e.$ Thus, $f = g$, $\upsilon - a.e.$ Then (f_n) converges to f in the normed space $\left(L^{p),\theta}(\mu), \|.\|\right)$. Then the norms $\|.\|$ and $\|.\|_{p),\theta,\mu}$ are equivalent (see proposition 11, in [18]), and so there exists a constant $C(p,q) > 0$ such that

$$\|f\| \leq C(p,q) \|f\|_{p),\theta,\mu}$$

for all $f \in L^{p),\theta}(\mu)$. This implies

$$\|f\|_{q),\theta,\upsilon} \leq \|f\| \leq C(p,q) \|f\|_{p),\theta,\mu}$$

for all $f \in L^{p),\theta}(\mu)$. On the other hand, by Lemma 1, μ and υ are absolutely continuous with respect to each other. This completes the proof. □

Theorem 2 *Let* $(\Omega, \sum, \mu)$ *and* $(\Omega, \sum, \upsilon)$ *be two finite measure spaces. Then the following statements are equivalent.*

1. *We have* $L^{p),\theta}(\mu) \subseteq L^{p),\theta}(\upsilon)$ *for* $p>1$ *and for all* $\theta \geq 0$.
2. $\mu \approx \upsilon$ *and there exists a constant* $C(p,\theta)>0$ *such that*

$$\sup_{0<\varepsilon\leq q-1} (\upsilon(E))^{\frac{1}{p-\varepsilon}} \leq C(p,\theta) \sup_{0<\varepsilon\leq p-1} (\mu(E))^{\frac{1}{p-\varepsilon}}$$

for all $E \in \sum$.

3. $L^1(\mu) \subseteq L^1(\upsilon)$.
4. $L^{p),\theta}(\mu) \subseteq L^{p),\theta}(\upsilon)$ *for* $p>1$ *and for all* $\theta \geq 0$.

Proof (1) $\Longrightarrow$ (2) : By Theorem 1, $\mu \approx \upsilon$ and there exists $C(p,\theta)>0$ such that

$$\|f\|_{p),\theta,\upsilon} \leq C(p,\theta) \|f\|_{p),\theta,\mu} \tag{3}$$

for all $f \in L^{p),\theta}(\mu)$. If $E \in \sum$, then $\chi_E \in L^p(\mu)$. Since $L^p(\mu) \subset L^{p),\theta}(\mu) \subset L^{p),\theta}(\upsilon)$, then $\chi_E \in L^{p),\theta}(\mu) \subset L^{p),\theta}(\upsilon)$ and by (3) we have

$$\|\chi_E\|_{p),\theta,\upsilon} \leq C(p,\theta) \|\chi_E\|_{p),\theta,\mu}. \tag{4}$$

Thus,

$$\sup_{0<\varepsilon\leq p-1} \left(\varepsilon^\theta \upsilon(E)\right)^{\frac{1}{p-\varepsilon}} \leq C(p,\theta) \sup_{0<\varepsilon\leq p-1} \left(\varepsilon^\theta \mu(E)\right)^{\frac{1}{p-\varepsilon}}. \tag{5}$$

(2) $\Longrightarrow$ (3) : Since when $\theta=0$, the space $L^{p),\theta}(\mu)$ reduces to the Lebesgue space $L^p(\mu)$, by (5),

$$(\upsilon(E))^{\frac{1}{p}} \leq C(p,0)(\mu(E))^{\frac{1}{p}} = C(p)(\mu(E))^{\frac{1}{p}}.$$

This implies

$$\upsilon(E) \leq M\mu(E), \tag{6}$$

where $M = C(p)^p$. Then by Proposition 1 in [13], we have $L^1(\mu) \subseteq L^1(\upsilon)$.

(3) $\Longrightarrow$ (4) : By the inclusion $L^1(\mu) \subseteq L^1(\upsilon)$ there exists $C_1>0$ such that

$$\|g\|_{1,\upsilon} \leq C_1 \|g\|_{1,\mu} \tag{7}$$

for all $g \in L^1(\mu)$. Let $f \in L^{p),\theta}(\mu)$. Then

$$\|f\|_{p),\theta,\mu} = \sup_{0<\varepsilon\leq p-1} \left(\varepsilon^\theta \int_\Omega |f|^{p-\varepsilon} d\mu\right)^{\frac{1}{p-\varepsilon}} < M$$

for some $M>0$. This implies $|f|^{p-\varepsilon} \in L^1(\mu)$ for all $\varepsilon \in (0,p-1]$. Since $L^1(\mu) \subseteq L^1(\upsilon)$, then $|f|^{p-\varepsilon} \in L^1(\upsilon)$. By (7) we have

$$\int_\Omega |f|^{p-\varepsilon} d\upsilon \leq C_1 \int_\Omega |f|^{p-\varepsilon} d\mu.$$

Thus, we obtain

$$\left(\int_{\Omega} |f|^{p-\varepsilon}\, d\upsilon\right)^{\frac{1}{p-\varepsilon}} \leq C \left(\int_{\Omega} |f|^{p-\varepsilon}\, d\mu\right)^{\frac{1}{p-\varepsilon}},$$

where $C = C_1^{\frac{1}{p-\varepsilon}}$. If we get the supremum in both sides, we have

$$\sup_{0<\varepsilon\leq p-1} \left(\varepsilon^{\theta} \int_{\Omega} |f|^{p-\varepsilon}\, d\upsilon\right)^{\frac{1}{p-\varepsilon}} \leq C \sup_{0<\varepsilon\leq p-1} \left(\varepsilon^{\theta} \int_{\Omega} |f|^{p-\varepsilon}\, d\mu\right)^{\frac{1}{p-\varepsilon}},$$

for all $\theta \geq 0$. Then

$$\|f\|_{p),\theta,\upsilon} \leq C \|f\|_{p),\theta,\mu} < CM < \infty$$

for all $f \in L^{p),\theta}(\mu)$. Finally, we have $L^{p),\theta}(\mu) \subseteq L^{p),\theta}(\upsilon)$ for all $\theta \geq 0$.

(4) $\implies$ (1) : This is easy. □

Theorem 3 *Let $(\Omega, \sum, \mu)$ be a finite measure space and let p and q be any two positive real numbers. Then*

$$L^{p),\theta}(\mu) \subseteq L^{q),\theta}(\mu) \tag{8}$$

whenever $1 < q < p$, and for all $\theta \geq 0$.

Proof Since for every $0 < \varepsilon \leq q-1$, we have $q - \varepsilon < p - \varepsilon$, then $L^{p-\varepsilon}(\mu) \subset L^{q-\varepsilon}(\mu)$. Thus, there exists $C > 0$ such that

$$\|f\|_{q-\varepsilon} \leq C \|f\|_{p-\varepsilon}$$

for all $f \in L^{p),\theta}(\mu)$. Let $f \in L^{p),\theta}(\mu)$. We have

$$\begin{aligned}
\|f\|_{q),\theta,\mu} &= \sup_{0<\varepsilon\leq q-1} \left(\varepsilon^{\theta} \int_{\Omega} |f|^{q-\varepsilon}\, d\mu\right)^{\frac{1}{q-\varepsilon}} = \sup_{0<\varepsilon\leq q-1} \varepsilon^{\frac{\theta}{q-\varepsilon}} \|f\|_{q-\varepsilon} \\
&\leq C \sup_{0<\varepsilon\leq q-1} \varepsilon^{\frac{\theta}{q-\varepsilon}} \|f\|_{p-\varepsilon} = C \sup_{0<\varepsilon\leq q-1} \varepsilon^{\frac{\theta}{q-\varepsilon}} \varepsilon^{\frac{\theta}{p-\varepsilon}} \varepsilon^{\frac{-\theta}{p-\varepsilon}} \|f\|_{p-\varepsilon} \\
&= C \sup_{0<\varepsilon\leq q-1} \varepsilon^{\frac{\theta(p-q)}{(p-\varepsilon)(q-\varepsilon)}} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{p-\varepsilon} \\
&\leq C \sup_{0<\varepsilon\leq q-1} \varepsilon^{\frac{\theta(p-q)}{(p-\varepsilon)(q-\varepsilon)}} \sup_{0<\varepsilon\leq q-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{p-\varepsilon} \\
&\leq C_0 \sup_{0<\varepsilon\leq p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{p-\varepsilon} = C_0 \|f\|_{p),\theta,\mu},
\end{aligned}$$

where $C_0 = C \sup_{0<\varepsilon\leq q-1} \varepsilon^{\frac{\theta(p-q)}{(p-\varepsilon)(q-\varepsilon)}}$. Since $q < p$, C_0 is finite and thus $f \in L^{q),\theta}(\mu)$. Hence,

$$L^{p),\theta}(\mu) \subseteq L^{q),\theta}(\mu)$$

whenever $p < q$, and for all $\theta \geq 0$. □

3. Approximate identities and consequences

In this section we will assume that Ω is a bounded subset of $\mathbb{R}^n$ and $1<p,q<\infty,\ \theta\geq 0.$

We know that $C_c^{\infty}(\Omega)$ is not dense in $L^{p),\theta}(\mu)$, where $C_c^{\infty}(\Omega)$ denotes the space of infinitely differentiable complex-valued functions with compact support on Ω [9]. Its closure $[L^p]_{p),\theta}$ consists of functions $f\in L^{p),\theta}(\mu)$ such that

$$\lim_{\varepsilon\to 0}\varepsilon^{\frac{\theta}{p-\varepsilon}}\|f\|_{p-\varepsilon}=0.$$

It is known that the Lebesgue space $L^p(\mu)$ admits an approximate identity bounded in $L^1(\mu)$ [5, 8]. The following theorem shows that the this property is not true for generalized grand Lebesgue space.

Theorem 4 *The generalized grand Lebesgue space $L^{p),\theta}(\mu)$ does not admit an approximate identity, bounded in $L^1(\mu)$.*

Proof Assume that $(e_\alpha)_{\alpha\in I}$ is an approximate identity in $L^{p),\theta}(\mu)$ bounded in $L^1(\mu)$. Then there exists a constant $M>0$ such that $\|e_\alpha\|_1<M$ for all $\alpha\in I$. Take any function $f\in L^{p),\theta}(\mu)-[L^p]_{p),\theta}$ (for example the function $f(t)=x^{-\frac{1}{p}},\ 1<p<\infty$). Then $e_\alpha * f\to f$ in $L^{p),\theta}(\mu)$. Since

$$\begin{aligned}\lim_{\varepsilon\to 0}\left(\varepsilon^{\theta}\int_{\Omega}|e_\alpha * f|^{p-\varepsilon}d\mu\right)^{\frac{1}{p-\varepsilon}} &= \lim_{\varepsilon\to 0}\varepsilon^{\frac{\theta}{p-\varepsilon}}\|e_\alpha * f\|_{p-\varepsilon}\\ &\leq \lim_{\varepsilon\to 0}\varepsilon^{\frac{\theta}{p-\varepsilon}}\|e_\alpha\|_1\|f\|_{p-\varepsilon}\\ &\leq M\lim_{\varepsilon\to 0}\varepsilon^{\frac{\theta}{p-\varepsilon}}\|f\|_{p-\varepsilon}=0,\end{aligned}$$

then $e_\alpha * f\in[L^p]_{p),\theta}$ for each $\alpha\in I$. This implies $f\in[L^p]_{p),\theta}$. This contradicts the assumption $f\in L^{p),\theta}(\mu)-[L^p]_{p),\theta}$. Then $L^{p),\theta}(\mu)$ does not admit an approximate identity bounded in $L^1(\mu)$. □

Theorem 5 *a. The generalized grand Lebesgue space $L^{p),\theta}(\mu)$ is a Banach convolution module over $L^1(\mu)$.*

b. The space $[L^p]_{p),\theta}$ is a Banach convolution module over $L^1(\mu)$.

Proof *a.* We know that $L^{p),\theta}(\mu)$ is a Banach space [9], and $L^p(\mu)$ is a Banach $L^1(\mu)-$module. Let $f\in L^1(\mu)$ and $g\in L^{p),\theta}(\mu)$. Then

$$\begin{aligned}\|f*g\|_{p),\theta} &= \sup_{0<\varepsilon\leq p-1}\left(\varepsilon^{\theta}\int_{\Omega}|f*g|^{p-\varepsilon}d\mu\right)^{\frac{1}{p-\varepsilon}} \qquad (9)\\ &= \sup_{0<\varepsilon\leq p-1}\varepsilon^{\frac{\theta}{p-\varepsilon}}\|f*g\|_{p-\varepsilon}\leq \sup_{0<\varepsilon\leq p-1}\varepsilon^{\frac{\theta}{p-\varepsilon}}\|f\|_1\|g\|_{p-\varepsilon}\\ &= \|f\|_1\sup_{0<\varepsilon\leq p-1}\varepsilon^{\frac{\theta}{p-\varepsilon}}\|g\|_{p-\varepsilon}=\|f\|_1\|g\|_{p),\theta}.\end{aligned}$$

It is easy to prove the other conditions for $L^{p),\theta}(\mu)$ to be a Banach convolution module over $L^1(\mu)$.

b. It is easy to see that $[L^p]_{p),\theta}$ is a vector space. Since $[L^p]_{p),\theta} \subset L^{p),\theta}(\mu)$ is closed in $L^{p),\theta}(\mu)$, and $L^{p),\theta}(\mu)$ is a Banach space, then $[L^p]_{p),\theta}$ is a Banach space. The inequality (9) is satisfied for all $f \in L^1(\mu)$ and $g \in [L^p]_{p),\theta}$. Then $[L^p]_{p),\theta}$ is a Banach $L^1(\mu)-$ module. □

Theorem 6 *a. The space $[L^p]_{p),\theta}$ admits an approximate identity bounded in $L^1(\mu)$.*

b. $[L^p]_{p),\theta}$ admits an approximate identity bounded in $L^1(\mu)$ and with compactly supported Fourier transforms.

Proof First we shall prove that the closure of $L^p(\mu)$ in $L^{p),\theta}(\mu)$ is $[L^p]_{p),\theta}$. Let $h \in L^p(\mu)$ be given. Since $L^p(\mu) \subset L^{p),\theta}(\mu) \subset L^{p-\varepsilon}(\mu)$, then

$$\lim_{\varepsilon\to 0}\left(\varepsilon^{\theta}\int_{\Omega}|h|^{p-\varepsilon}\,d\mu\right)^{\frac{1}{p-\varepsilon}} = \lim_{\varepsilon\to 0}\varepsilon^{\frac{\theta}{p-\varepsilon}}\,\|h\|_{p-\varepsilon} = 0.$$

Hence, $h \in [L^p]_{p),\theta}$. This implies

$$L^p(\mu) \subset [L^p]_{p),\theta}.$$

Since

$$C_c^{\infty}(\mathbb{R}^n) \subset L^p(\mu) \subset [L^p]_{p),\theta}, \tag{10}$$

we have

$$[L^p]_{p),\theta} = \overline{C_c^{\infty}(\mathbb{R}^n)} \subset \overline{L^p(\mu)} \subset [L^p]_{p),\theta},$$

where the closures are in the norm $\|.\|_{p),\theta,\mu}$. Then

$$\overline{L^p(\mu)} = \overline{C_c^{\infty}(\mathbb{R}^n)} = [L^p]_{p),\theta}. \tag{11}$$

It is known by Lemma 1.12 in [8] that $L^p(\mu)$ admits an approximate identity $(e)_{\alpha\in I}$, bounded in $L^1(\mu)$. Then there exists a constant $M > 1$, such that $\|e_\alpha\|_1 \le M$ for all $\alpha \in I$. Also, given any $u \in L^p(\mu)$ and $\delta > 0$, there exists $\alpha_0 \in I$ such that

$$\|e_\alpha * u - u\|_p \le \frac{\delta}{3} \tag{12}$$

for all $\alpha \ge \alpha_0$. We shall show that $(e)_{\alpha\in I}$ is also an approximate identity in $[L^p]_{p),\theta}$. Let $f \in [L^p]_{p),\theta}$ be given. Since $L^p(\mu)$ is dense in $[L^p]_{p),\theta}$, in the norm $\|.\|_{p),\theta}$, there exists $g \in L^p(\mu)$ such that

$$\|f-g\|_{p),\theta} \le \frac{\delta}{3M}. \tag{13}$$

Then

$$\begin{aligned}\|e_\alpha * f - f\|_{p),\theta} &= \|e_\alpha * f - f - e_\alpha * g + e_\alpha * g + g - g\|_{p),\theta} \\ &\le \|e_\alpha * f - e_\alpha * g\|_{p),\theta} + \|e_\alpha * g - g\|_{p),\theta} + \|g-f\|_{p),\theta},\end{aligned} \tag{14}$$

and

$$\|e_\alpha * f - e_\alpha * g\|_{p),\theta} = \|e_\alpha * (f-g)\|_{p),\theta} \qquad (15)$$
$$\leq \|e_\alpha\|_1 \|(f-g)\|_{p),\theta} \leq M \|(f-g)\|_{p),\theta} \leq M\frac{\delta}{3M} = \frac{\delta}{3}.$$

Since $M > 1,$ combining (12) $(13), (14),$ and $(15),$ we obtain

$$\|e_\alpha * f - f\|_{p),\theta} \leq \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3M} < \delta.$$

This completes the proof of part (a). The proof of part (b) is obvious. □

As an application of the approximate identities we will give the following theorem.

Theorem 7 *a) The space of multipliers* $M\left(L^1(\mu), ([L^p]_{p),\theta})^*\right)$ *is isometrically isomorphic to dual space* $([L^p]_{p),\theta})^*$ *(dual of* $[L^p]_{p),\theta}$*).*

b) The space of multipliers $M\left(L^1(\mu), \left(L^{p),\theta}(\mu)\right)^*\right)$ *is isometrically isomorphic to the dual space* $\left(L^1(\mu) * L^{p),\theta}(\mu)\right)^*$*. If* f *is an element in the space of multipliers* $M\left(L^1(\mu), \left(L^{p),\theta}(\mu)\right)^*\right)$*, then there is an extension* F *of* f *to a continuous linear form on* $L^{p),\theta}(\mu)$ *so that*

$$\left\|F \mid \left(L^{p),\theta}(\mu)\right)^*\right\| = \left\|f \mid \left(L^1(\mu) * L^{p),\theta}(\mu)\right)^*\right\|,$$

where $\left\|F \mid \left(L^{p),\theta}(\mu)\right)^*\right\|$ *and* $\left\|f \mid \left(L^1(\mu) * L^{p),\theta}(\mu)\right)^*\right\|$ *denote the norms on the spaces* $\left(L^{p),\theta}(\mu)\right)^*$ *and* $\left(L^1(\mu) * L^{p),\theta}(\mu)\right)^*$*, respectively.*

Proof $a)$ We know by Theorem 5 that $[L^p]_{p),\theta}$ is a Banach $L^1(\mu)-$module. Also, by Theorem 6, $L^1(\mu) * [L^p]_{p),\theta}$ is dense in $[L^p]_{p),\theta}$ in the $\|.\|_{p),\theta,\mu}$ norm. Then by the module factorization theorem [20], we have

$$L^1(\mu) * [L^p]_{p),\theta} = [L^p]_{p),\theta}. \qquad (16)$$

Thus, $[L^p]_{p),\theta}$ is an essential Banach module over $L^1(\mu)$. Then by Corollary 2.13 in [15], and by (16) we obtain

$$M\left(L^1(\mu), ([L^p]_{p),\theta})^*\right) = \left(L^1(\mu) * [L^p]_{p),\theta}\right)^* = ([L^p]_{p),\theta})^*.$$

$b)$ Again by Corollary 2.13 in [15],

$$M\left(L^1(\mu), \left(L^{p),\theta}(\mu)\right)^*\right) = \left(L^1(\mu) * L^{p),\theta}(\mu)\right)^*.$$

On the other hand, by Theorem 5, $L^{p),\theta}(\mu)$ is a Banach $L^1(\mu)-$convolution module. Thus, $L^1(\mu) * L^{p),\theta}(\mu) \subset L^{p),\theta}(\mu)$. Then if $f \in M\left(L^1(\mu), \left(L^{p),\theta}(\mu)\right)^*\right)$, by the Hahn–Banach extension theorem, there is an extension F of f to a continuous linear form on $L^{p),\theta}(\mu)$ so that $\left\|F \mid \left(L^{p),\theta}(\mu)\right)^*\right\| = \left\|f \mid \left(L^1(\mu) * L^{p),\theta}(\mu)\right)^*\right\|$. This completes the proof. □

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